A Discrete Queue, Fourier Sampling on Szegö Curves, and Spitzer Formulas

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Abstract

We consider a discrete-time multi-server queue for which the moments of the stationary queue length can be expressed in terms of series over the zeros in the closed unit disk of a queue-specific characteristic function. In many important cases these zeros can be considered to be located on a queue-specific curve, called generalized Szegö curve. By adopting a special parametrization of these Szegö curves, the relevant zeros occur as equidistant samples of a 2π-periodic function whose Fourier coefficients can be determined analytically. Thus the series occurring in the expressions for the moments can be written as Fourier aliasing series with terms given in analytic form. This gives rise to formulas for e.g. the mean and variance of the queue length that are reminiscent of Spitzer's identity for the moment generating function of the steady-state waiting time for a $G/G/1$ queue. Indeed, by considering the queue under investigation as a $G/G/1$ queue, the new formulas for the mean and variance also follow from Spitzer's identity. The approach in this paper can also be used to compute the probability distribution function of the queue length in analytic form.

Keywords: Discrete-time queue, multi-server, Szegö curve, Spitzer's identity, Fourier sampling.

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1 Introduction and motivation

We consider a discrete-time queue (in queueing theory language a multi-server queue, see [6]), defined by the recursion

$$X_{n+1} = \max\{X_n - s, 0\} + A_n \ , \qquad n \in \mathbb{Z} \ . \tag{1.1}$$

Here X_n is the queue length at the beginning of time slot n, A_n is a non-negative discrete random variable denoting the number of arriving customers at the end of slot n, and s is the (constant) number of customers that can be processed within one time slot. It is assumed that the A_n form an i.i.d. sequence of random variables with probabilities $a_j = P(A = j)$, $j = 0, 1, \ldots$, such that $\sum_j a_j = 1$ and

$$E(A) = \sum_{j=0}^{\infty} j\, a_j < s\ . \tag{1.2}$$

Under this assumption the system defined by (1.1) is stable and the stationary probability distribution of the X_n exists. We denote this stationary distribution by X with probabilities $x_j = P(X = j) = \lim_{n\to\infty} P(X_n = j)$, $j = 0, 1, \ldots$, satisfying $\sum_j x_j = 1$. We shall also assume that the generating function

$$A(z) := \sum_{j=0}^{\infty} a_j\, z^j \tag{1.3}$$

is analytic in a disk $|z| < 1 + \varepsilon$ with $\varepsilon > 0$. Then the generating function

$$X(z) := \sum_{j=0}^{\infty} x_j\, z^j \tag{1.4}$$

is analytic in a disk $|z| < 1+\varepsilon$ with $\varepsilon > 0$ as well, and it is an elementary exercise in queueing theory to show that the generating functions $A(z)$ and $X(z)$ satisfy (see e.g. [2])

$$X(z) = \frac{A(z)\ \sum_{j=0}^{s-1} (z^s - z^j)\, x_j}{z^s - A(z)} \tag{1.5}$$

in a disk $|z| < 1 + \varepsilon$ with $\varepsilon > 0$. We refer to [7] for the general theory of Markov chains (of which the system in (1.1) is an example), and to [4] for the theory and applications of queueing systems. In Sec. 7 we shall relate the queueing system under investigation to what is called in queueing theory a $G/G/1$ queue.

It follows from Rouché's theorem, applied to $z^s - A(z)$ on circles $|z| = 1 + \varepsilon$ with $\varepsilon > 0$, and the assumption (1.2), that the equation

$$A(z) = z^s \tag{1.6}$$

has exactly s roots $z_0 = 1, z_1, \ldots, z_{s-1}$ in $|z| \le 1$. Also, there is an $\varepsilon > 0$ such that (1.6) has no roots for $1 < |z| < 1 + \varepsilon$. Denote the mean and variance of A by μ_A and σ_A^2, so that

$$\mu_A = \sum_{j=0}^{\infty} j\, a_j = A'(1)\ , \tag{1.7}$$

$$\sigma_A^2 = \sum_{j=0}^{\infty} (j - \mu_A)^2\, a_j = A''(1) + A'(1) - (A'(1))^2\ , \tag{1.8}$$

and similarly for X. Now a careful but otherwise elementary analysis of the relation (1.5) around $z = 1$, using the analyticity of $A(z)$ and $X(z)$ in a disk $|z| < 1 + \varepsilon$ with $\varepsilon > 0$, yields

$$\mu_X = \frac{1}{2}\mu_A + \frac{\sigma_A^2}{2(s-\mu_A)} - \frac{1}{2}(s-1) + \sum_{k=1}^{s-1} \frac{1}{1-z_k} \ , \tag{1.9}$$

$$\begin{aligned}\sigma_X^2 = \sigma_A^2 \ &+ \ \frac{A'''(1) - s(s-1)(s-2)}{3(s-\mu_A)} + \frac{A'' - s(s-1)}{2(s-\mu_A)} \\ &+ \ \Big(\frac{A''(1) - s(s-1)}{2(s-\mu_A)}\Big)^2 - \sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \ .\end{aligned} \tag{1.10}$$

See e.g. [2], and also the beginning of Sec. 6 for some aspects of this analysis.

The two series at the right-hand sides of (1.9) and (1.10) can be evaluated by numerically computing the zeros z_k, $k = 1, \ldots, s$. The feasibility of this approach depends on A and how large s is. In [5] these two series are bounded in a relatively simple form in terms of the first three moments of A. This gives considerable insight into the behaviour of these series, but the bounds are not always as tight as one wishes. As an alternative, one can try to consider the z_k's as equidistant samples $z(2\pi k/s)$ of a 2π-periodic, complex-valued function parametrizing the "curve" $\{z \mid |z| \leq 1, \ |A(z)|^{1/s} = |z|\}$, and apply Fourier sampling theory. In this paper we work out this point of view in all detail, and we succeed in obtaining analytical expressions for the two series. The resulting formulas for μ_X and σ_X^2, see formulas (3.6-3.7) below, are reminiscent of the expressions for the mean and variance of the steady-state waiting time for a general $G/G/1$ queue, see [1], (8). These formulas explicitly involve the power series coefficients of $A^l(z)$, $l = 0, 1, \ldots$, around $z = 0$, and are therefore termed formulas of Spitzer type, since they follow from Spitzer's identity, see [15], [1], (7), and (3.9) below.

That methods from analytic function theory play a crucial role in queueing theory is evident, notably from the work of Pollaczek [10, 11], also see [1, 14]. In [1] it is pointed out that Spitzer's identity can be derived from one of Pollaczek's identities, see [1], (3), and this bridges a gap between the analytic function theory approach and the combinatorial approach as embodied by Spitzer's identity.

In the present paper we fully exploit the discrete nature of the queues under study, and we bridge the gap between the analytic function approach, as embodied by the formulas (1.9-1.10), and formulas of Spitzer type, by considering the relevant zeros as sample points on, what we call generalized Szegö curves. This approach yields the desired analytic expressions for the two series as well as for the probabilities x_j, $j = 0, ..., s-1$, that occur at the right-hand side of (1.5), and x_s. From x_j, $j = 0, 1, ..., s$, all x_j with $j = s+1, s+2, ...$ can be determined recursively using (1.5). In [8] we compare the analytic method thus obtained for computing the probabilities x_j to methods with a more numerical character.

2 Overview

We now sketch our approach to obtain analytic expressions for series $\sum_k g(z_k)$ with z_k the roots of $A(z) = z^s$ in $|z| \leq 1$. We shall throughout assume that $a_0 > 0$. This involves no essential limitation: if a_0 were zero we would replace the distribution $\{a_i\}$ by $\{a_i'\}$ where

$a'_i = a_{i-m}$, a_m being the first non-zero entry of $\{a_i\}$, and a corresponding decrease in the maximum number of customers served per slot according to $s' = s - m$.

We consider for w in a neighbourhood of 0 the equation

$$z\,A^{-1/s}(z) = w \ , \tag{2.1}$$

where at the left-hand side of (2.1) we have taken the principal value of the root. Let g be analytic in a neighbourhood of $z = 0$. By the Lagrange inversion theorem, see [17], § 7.32, there is a neighbourhood of $w = 0$ such that the equation (2.1) has a unique solution $z = z_0(w)$. Furthermore, the function $g(z_0(w))$ has the power series expansion

$$g(z_0(w)) = g(0) + \sum_{l=1}^{\infty} c_l(g)\,w^l \ , \tag{2.2}$$

where for $l = 1, 2, ...$

$$c_l(g) = \frac{1}{l!}\left(\frac{d}{dz}\right)^{l-1} [A^{l/s}(z)\,g'(z)]\,(z=0) = \frac{1}{l}\,C_{z^{l-1}}[A^{l/s}(z)\,g'(z)] \ . \tag{2.3}$$

We have used here the short-hand notation $C_{z^j}[f(z)]$ for the coefficient of z^j in $f(z)$. We denote

$$c_l := c_l(g_0) \ ; \qquad g_0(z) = z \ , \tag{2.4}$$

and we let R be the radius of convergence of the series

$$z_0(w) = \sum_{l=1}^{\infty} c_l\,w^l \ . \tag{2.5}$$

We shall show in Sec. 4 that the mapping w, $|w| < R \rightarrow z_0(w)$ is analytic and injective.

Now assume that $R > 1$. Then we can consider the equation (2.1) and its unique solution $z_0(w)$ with $w = e^{i\alpha}$, $\alpha \in [0, 2\pi]$. Accordingly, we let

$$z(\alpha) := z_0(e^{i\alpha}) \ , \qquad \alpha \in [0, 2\pi] \ . \tag{2.6}$$

The s roots $z = z_k$, $k = 0, 1, ..., s-1$, of the equation $A(z) = z^s$ with $z_0 = 1$, $|z_k| \leq 1$, $k = 1, ..., s-1$, are distinct and are obtained as

$$z_k = z(2\pi k/s) = z_0(e^{2\pi ik/s}) \ , \qquad k = 0, 1, ..., s-1 \ . \tag{2.7}$$

Furthermore, with (2.6) we have a parametrization of a Jordan curve with 0 in its interior. Finally, when g is analytic in an open neighbourhood of $\{z_0(w)\,|\,|w| \leq 1\}$, then the 2π-periodic function $\alpha \rightarrow g(z(\alpha))$ has the Fourier series representation

$$g(z(\alpha)) = g(0) + \sum_{l=1}^{\infty} c_l(g)\,e^{il\alpha} \ , \qquad \alpha \in [0, 2\pi] \ , \tag{2.8}$$

with $c_l(g)$ given in (2.3).

The assumption $R > 1$ is, for instance, satisfied when $A(z)$ is zero-free in $|z| \leq 1$. An example of this is the Poisson case,

$$a_j = e^{-\lambda}\,\frac{\lambda^j}{j!} \ , \quad j = 0, 1, ... \ ; \qquad A(z) = e^{\lambda(z-1)} \ , \tag{2.9}$$

with $0 \leq E(A) = \lambda < s$. There are also non-trivial examples of distributions A with generating functions that do have zeros in the unit disk for which $R > 1$. See Example 4.5 at the end of Sec. 4 where we consider the binomial distribution

$$a_j = \binom{n}{j} q^j (1-q)^{n-j},\ j = 0, ..., n; \quad a_j = 0,\ j = n+1, ... \ , \tag{2.10}$$

so that

$$A(z) = (1 - q + qz)^n \ , \tag{2.11}$$

with $E(A) = nq < s$.

With $R > 1$ and g analytic in an open neighbourhood of $\{z_0(w) \mid |w| \leq 1\}$ there follows from the Fourier series representation (2.8) and elementary Fourier sampling theory that

$$\sum_{k=0}^{s-1} g(z_k) = \sum_{k=0}^{s-1} g(z(2\pi k/s)) = s\, g(0) + s \sum_{l=1}^{\infty} c_{ls}(g) \ , \tag{2.12}$$

with $c_l(g)$ given by (2.3). Thus

$$\sum_{k=0}^{s-1} g(z_k) = s\, g(0) + \sum_{l=1}^{\infty} \frac{1}{l}\, C_{z^{ls-1}}[A^l(z)\, g'(z)] \ . \tag{2.13}$$

Note that the right-hand side series in (2.13) has terms that involve integral powers of A only. In fact, when g is analytic in a disk $|z| < 1+\varepsilon$ with $\varepsilon > 0$ and A satisfies the assumptions of Sec. 1, then the numbers $C_{z^{ls-1}}[A^l(z)\, g'(z)]$ decay exponentially fast, irrespective whether $R > 1$ or not. Hence in these cases the right-hand side of (2.13) makes sense regardless whether $R > 1$ or not. It therefore seems a plausible conjecture that (2.13) holds for these more general A and somewhat different type of g.

Some of the g's we are interested in fail to be analytic at $z = 1$, but become so after proper regularization. This is, for instance, the case with

$$g(z) = \frac{1}{1-z}\ ,\ \frac{z}{(1-z)^2} \tag{2.14}$$

that occur in (1.9) and (1.10). In Sec. 5 we regularize the g's in (2.14) by subtracting

$$\frac{B}{1 - z\, A^{-1/s}(z)}\ ,\quad \frac{C}{(1 - z\, A^{-1/s}(z))^2} + \frac{D}{1 - z\, A^{-1/s}(z)} \tag{2.15}$$

with properly chosen B and C, D. Indeed, since $A(1) = 1$, proper choice of B and C, D cancels the poles of the g's at $z = 1$. Furthermore

$$z\, A^{-1/s}(z)|_{z=z_k} = e^{2\pi i k/s}\ , \qquad k = 1, ..., s\ ,$$

and in Sec. 5 we present an identity for the series $\sum_{k=1}^{s-1} (1 - e^{2\pi i k/s})^{-m}$, $m = 1, 2$, which shows that regularization of the g's in (2.14) according to (2.15) may maintain the analytic nature of the expressions for $\sum g(z_k)$. For this it is also required that the c_l, with regularized g's, are still expressible in analytic form. That this happens to be the case is also shown in Sec. 5.

We now give a short survey of the paper. In Sec. 3 we present the main results of this paper. In Sec. 4 we give the details regarding the parametrization in (2.5) of the Jordan curve $\{z \mid |z| = |A^{1/s}(z)|,\ |z| \le 1\}$ in case that $R > 1$. We call curves of this type generalized Szegö curves, the curve $\{z \mid |z| = |e^{\vartheta(z-1)}|,\ |z| \le 1\}$ as considered in [15] with $\vartheta = 1$ being the prototype of these curves. In Sec. 5 we give the details of our approach to find analytic expressions for $\sum_{k=1}^{s-1}(1-z_k)^{-1}$, $\sum_{k=1}^{s-1} z_k(1-z_k)^{-2}$, and we present the resulting expressions for μ_X and σ_X^2. In Sec. 6 we give explicit expressions for the probabilities x_j, $j = 0, 1, ..., s$, by using the approach of this paper. In Sec. 7 we view the queue in (1.1) as a $G/G/1$ queue, and we present Spitzer's formula for the moment generating function of the steady-state waiting time for this case. This connection yields alternative proofs of the results in Secs. 5, 6 without the assumption that $R > 1$. Hence, this completes the process of bridging a gap between two sets of formulas that exist for the mean and variance of the waiting times in certain discrete-time queues.

3 Results

We have the following main results, using the short-hand notation $C_{z^j}[f(z)]$ for the coefficient of z^j in $f(z)$.

Theorem 3.1. *Under the assumptions on A as made in Sec. 1 there holds*

$$\sum_{k=1}^{s-1} \frac{1}{1-z_k} = \frac{1}{2}(s-1) + \frac{1}{2}\mu_A - \frac{\sigma_A^2}{2(s-\mu_A)} + \sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=ls}^{\infty} (j-ls)\, C_{z^j}[A^l(z)]\ . \tag{3.1}$$

Theorem 3.2. *Under the assumptions on A as made in Sec. 1 there holds*

$$\begin{aligned}\sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} &= -\frac{1}{12}C(s-1)(s-5) + \frac{1}{2}D(s-1) - g_2^R(1) - s(C+D) \\ &\quad - \sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=ls}^{\infty} (j-ls)^2\, C_{z^j}[A^l(z)]\ ,\end{aligned} \tag{3.2}$$

where C and D are given by

$$C = \Big(1 - \frac{1}{s}\,\mu_A\Big)^2\ , \qquad C + D = \frac{1}{s}\,\sigma_A^2\ , \tag{3.3}$$

and

$$g_2^R(1) = \frac{a^{[2]} + \frac{1}{3}a^{[3]}}{1 + a^{[1]}} + \frac{a^{[1]} + \frac{1}{2}a^{[2]}}{1 + a^{[1]}}\Big(1 - \frac{a^{[1]} + \frac{1}{2}a^{[2]}}{1 + a^{[1]}}\Big) \tag{3.4}$$

with $a^{[i]}$ the i^{th} derivative of $A^{-1/s}(z)$ at $z = 1$, $i = 1, 2, 3$. Alternatively, one has for the constant on the first line of (3.2)

$$\begin{aligned}&-\frac{1}{12}C(s-1)(s-5) + \frac{1}{2}D(s-1) - g_2^R(1) - s(C+D) \\ &= \frac{A'''(1) - s(s-1)(s-2)}{3(s-\mu_A)} + \frac{A''(1) - s(s-1)}{2(s-\mu_A)} + \Big(\frac{A'' - s(s-1)}{2(s-\mu_A)}\Big)^2\ .\end{aligned} \tag{3.5}$$

Theorems 3.1-3.2 are proved in Sec. 5 under the assumption that $R > 1$, where R is the radius of convergence of the power series of $z_0(w)$ in (2.5). In Sec. 4 we shall show that $C_{z^j}[A^l(z)]$, $j \geq ls$, can be estimated in such a way that the two infinite series at the right-hand sides of (3.1) and (3.2) converge absolutely under the assumptions on A of Sec. 1 alone (no assumption on R required). A further observation is that from Thms. 3.1-3.2 and (1.9), (1.10)

$$\mu_X = \mu_A + \sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=ls}^{\infty} (j - ls)\, C_{z^j}[A^l(z)] \ , \tag{3.6}$$

$$\sigma_X^2 = \sigma_A^2 + \sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=ls}^{\infty} (j - ls)^2\, C_{z^j}[A^l(z)] \ . \tag{3.7}$$

These two results can be proved directly, under the assumptions on A of Sec. 1, by using Spitzer's identity, see [13] and [4], p.339. To that end, we introduce the process

$$W_{t+1} = \max(W_t + A_{t-1} - s, 0) \ , \tag{3.8}$$

and W its stationary distribution, i.e. $P(W = j) = \lim_{t\to\infty} P(W_t = j)$. Observe that from (1.1) it follows that $X_t = W_t + A_{t-1}$. Spitzer's identity now reads

$$E(e^{-uW}) = \exp\Big\{\sum_{l=1}^{\infty} \frac{1}{l}\, E(e^{-u\,\max(S_l,0)} - 1)\Big\}, \quad \mathrm{Re}\ u \geq 0 \ , \tag{3.9}$$

where $S_l = \sum_{i=1}^{l} (A_i - s)$. This will be detailed in Sec. 7.

In Sec. 6 we consider the stationary queue length distribution $\{x_i\}$, for which we have the following result.

Theorem 3.3. *Under the assumptions on A as made in Sec. 1 there holds*

$$c := \sum_{j=0}^{s-1} x_j = \exp\Big\{-\sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=ls}^{\infty} C_{z^j}[A^l(z)]\Big\} \ , \tag{3.10}$$

$$d := \sum_{j=0}^{s} x_j = \exp\Big\{-\sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=ls+1}^{\infty} C_{z^j}[A^l(z)]\Big\} \ , \tag{3.11}$$

$$x_i = d\, C_{v^i}\Big[A(v)\, \exp\Big\{\sum_{j=1}^{s-1} v^j \sum_{l=1}^{\infty} \frac{1}{l}\, C_{z^{ls+j}}[A^l(z)]\Big\}\Big] \ , \tag{3.12}$$

for $i = 0, 1, ..., s - 1$. Hence, and in particular,

$$x_0 \ = \ a_0 d \ = \ a_0 \exp\Big\{-\sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=ls+1}^{\infty} C_{z^j}[A^l(z)]\Big\} \ , \tag{3.13}$$

$$x_s \ = \ d - c \ = \ d\Big(1 - \exp\Big\{-\sum_{l=1}^{\infty} \frac{1}{l}\, C_{z^{ls}}[A^l(z)]\Big\}\Big) \ . \tag{3.14}$$

From the x_j, $j = 0, 1, ..., s$, all other x_j's can be computed recursively, for it follows from (1.5) that

$$X(z) - c\,A(z) = A(z) \sum_{j=0}^{\infty} x_{j+s}\, z^j \tag{3.15}$$

with c given in (3.10).

As with Thms. 3.1-3.2 the proof of Thm. 3.3 is first given under the assumption that the radius of convergence R of the series in (2.5) is larger than one. This latter assumption can, however, be removed by directly using Spitzer's identity.

4 Fourier sampling on Szegö curves

In 1924, Szegö [15] showed that the zeros of the normalized partial sums

$$s_n(nz) = \sum_{k=0}^{n} \frac{(nz)^k}{k!} \ , \qquad n = 0, 1, ... \ , \tag{4.1}$$

of e^z tend to what nowadays is called the Szegö curve

$$\mathcal{S} := \{z \in \mathbb{C} \mid |z| = |e^{z-1}|, \ |z| \le 1\} \ . \tag{4.2}$$

This Szegö curve attracts attention to this date of researchers in approximation theory, see e.g. [12, 16, 18] and the references therein.

Curves of the Szegö type occur in the present context as follows. When $A(z)$ is as in Sec. 1, the roots z_k of $A(z) = z^s$ in the unit disk all lie in the set

$$\mathcal{S}_{A,s} := \{z \in \mathbb{C} \mid |z| = |A^{1/s}(z)|, \ |z| \le 1\} \ . \tag{4.3}$$

In the case that $A(z) = \exp(\lambda(z-1))$, the generating function of a Poisson distribution, with $A'(1) = \lambda < s$, we get the set

$$\mathcal{S}_\vartheta := \{z \in \mathbb{C} \mid |z| = |e^{\vartheta(z-1)}|, \ |z| \le 1\} \ , \tag{4.4}$$

where $\vartheta := \lambda/s$. Interestingly, in this case some of the quantities that occur in Sec. 3, such as d of (3.11), can be expressed in terms of the $s_n(nz)$ in (4.1). The set $\mathcal{S}$ in (4.2) occurs as the limit case where $\vartheta = 1$.

We start this section by proving the claims on the mappings $z_0(w)$ and $z(\alpha)$ made in Sec. 2 under the assumption that the power series

$$\sum_{l=1}^{\infty} c_l w^l \ ; \qquad c_l = \frac{1}{l}\, C_{z^{l-1}}[A^{l/s}(z)] \ , \tag{4.5}$$

has radius of convergence $R > 1$. We have assumed that $A(z)$ is analytic in a disk $|z| < 1+\varepsilon$, and also that $a_j \ge 0$, $A'(1) < s$, and $a_0 > 0$. Let $\varepsilon > 0$ be such that $A(z)$ is analytic in $|z| < 1+\varepsilon$. Let

$$G_\varepsilon := \Big\{\sum_{l=1}^{\infty} c_l w^l \mid |w| < R\Big\} \cap \{z \mid |z| < 1+\varepsilon\} \ . \tag{4.6}$$

Define

$$z_0(w) := \sum_{l=1}^{\infty} c_l w^l \ , \qquad |w| < R \ , \tag{4.7}$$

and let

$$H_\varepsilon := z_0^{\leftarrow}(G_\varepsilon) = \{w \in \mathbb{C} \mid |w| < R,\ |z_0(w)| < 1 + \varepsilon\} \ .$$

Lemma 4.1. *With the above assumptions and definitions the following holds. The function A is analytic and zero-free on G_ε. Taking the principal s^{-1}-root of $A(z)$, $z \in G_\varepsilon$, there holds*

$$z_0(w)\, A^{-1/s}(z_0(w)) = w \ , \qquad w \in H_\varepsilon \ , \tag{4.8}$$

and $z_0(w)$ is the unique solution of the equation $z\, A^{-1/s}(z) = w$ with $w \in H_\varepsilon$ and $z \in G_\varepsilon$. This unique $z_0(w)$ is positive for $w \in (0,1]$ and satisfies $z_0(1) = 1$. For $\alpha \in [0, 2\pi]$ we have that $z(\alpha)$ is the unique solution z in $|z| \le 1$ of

$$z\, A^{-1/s}(z) = e^{i\alpha} \ . \tag{4.9}$$

The set $\{z(\alpha) \mid \alpha \in [0, 2\pi]\}$ is a Jordan curve with 0 in its interior. Finally, the roots z_k of the equation $A(z) = z^s$, $k = 0, 1, ..., s-1$, occur as $z(2\pi k/s)$ and are distinct.

Proof. Evidently, A is analytic on $G_\varepsilon \subset \{z \mid |z| < 1 + \varepsilon\}$. Since $z_0(w)\, A^{-1/s}$ $(z_0(w)) = w$ holds in a neighbourhood of $w = 0$, we have by analyticity that

$$z_0^s(w) = w^s\, A(z_0(w)) \ , \qquad w \in H_\varepsilon \ . \tag{4.10}$$

Suppose that $w \in H_\varepsilon$, $w \neq 0$, and that $A(z_0(w)) = 0$. Then it follows from (4.10) that $z_0(w) = 0$, whence that $A(0) = 0 = a_0 \neq 0$. Contradiction. So $A(z_0(w)) \neq 0$ for $w \in H_\varepsilon$. We can therefore take the principal s^{-1}-root of $A(z)$ for $z \in G_\varepsilon$ which is analytic on G_ε. By analyticity we then have that $z_0(w)\, A^{-1/s}(z_0(w)) = w$ holds on all of H_ε, and not just in a neighbourhood of $w = 0$. That is, (4.8) holds. From (4.8) it readily follows that z_0 is injective on H_ε. Also when $w \in H_\varepsilon$ we have that $z_0(w)$ is the unique solution $z \in G_\varepsilon$ of the equation $z\, A^{-1/s}(z) = w$.

The function $z \in [0, 1+\delta] \to z\, A^{-1/s}(z)$ is strictly increasing for some $\delta > 0$. Indeed, when $z \in (0,1]$ we have that

$$\begin{aligned}(z\, A^{-1/s}(z))' &= \frac{1}{s}\, A^{-\frac{1}{s}-1}(z)\, [s\, A(z) - z\, A'(z)] \\ &= \frac{1}{s}\, A^{-\frac{1}{s}-1}(z) \sum_{j=0}^{\infty} (s-j)\, a_j\, z^j \ge \frac{1}{s}\, A^{-\frac{1}{s}-1}(z)\, z^s \sum_{j=0}^{\infty} (s-j)\, a_j > 0 \ ,\end{aligned} \tag{4.11}$$

since $A'(1) < s$. Moreover $A(1) = 1$. It thus follows that $z_0(w)$ increases from 0 to 1 as w increases from 0 to 1.

We consider now $w = e^{i\alpha}$ with $\alpha \in [0, 2\pi]$, and let $z(\alpha)$ be the unique solution of (4.9). We shall show that $|z(\alpha)| \le 1$. To that end we observe that there is a $\delta > 0$ such that $|A(z)| \neq |z|^s$ when $1 < |z| < 1 + \varepsilon$ (this follows from the assumptions that $a_j \ge 0$, $A'(1) < s$). Since $z(0) = z_0(1) = 1$ and $z(\alpha)$ depends continuously on $\alpha \in [0, 2\pi]$ we see that $|z(\alpha)| \le 1$, $\alpha \in [0, 2\pi]$, indeed. Furthermore, $z(0) = z(2\pi)$ and $z(\alpha) \neq z(\beta)$ when $0 \le \alpha < \beta < 2\pi$, while the mapping $r \in [0,1] \to \{z_0(r\,e^{i\alpha}) \mid \alpha \in [0,2\pi]\}$ is a homotopy between $\{0\}$ and $\{z(\alpha) \mid \alpha \in [0,2\pi]\}$. Hence $\{z(\alpha) \mid \alpha \in [0,2\pi]\}$ is indeed a Jordan curve with 0 in its interior.

Finally consider (4.9) with $\alpha = 2\pi k/s$, $k = 0,1,...,s-1$. Evidently, the $z(2\pi k/s)$ are distinct and have modulus ≤ 1, as follows from the above. Also, any $z(2\pi k/s)$ is a root of the equation $A(z) = z^s$, see (4.9). Hence, the sets $\{z_k \mid k = 0,...,s-1\}$ and $\{z(2\pi k/s) \mid k = 0,...,s-1\}$ coincide. □

Note. 1. We have $|A(z)|^{1/s} >,=,< |z|$ according as z, $|z| \leq 1$, is inside, on, outside the Jordan curve $\{z(\alpha) \mid \alpha \in [0,2\pi]\}$.

2. As is implicit in the proof of Lemma 4.1 we have the following geometric condition which is equivalent with $R > 1$: There is a Jordan curve J with $\mathcal{S}_{A,s}$ in its interior such that $A(z)$ is zero-free on and inside J while $|A(z)| < |z|^s$ on J. Also see [8], Subsec. 3.1 for this matter.

Lemma 4.2. *Assume that A satisfies the conditions in Sec. 1 and that $A(z)$ is zero-free in $|z| < 1+\varepsilon$, where $\varepsilon > 0$. Then $C_{z^{l-1}}[A^{l/s}(z)]$ decays exponentially.*

Proof. We have by Cauchy's theorem

$$C_{z^{l-1}}[A^{l/s}(z)] = \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{A^{l/s}(z)}{z^l}\,dz\ , \qquad l = 1,2,...\ , \tag{4.12}$$

for any $r \in (0,1+\varepsilon)$. Noting that there is a $\delta > 0$ such that

$$\Big|\frac{A(z)}{z^s}\Big| \leq \frac{A(|z|)}{|z|^s} < 1\ , \qquad 1 < |z| < 1+\delta\ , \tag{4.13}$$

we see that for any $r \in (1,1+\delta)$

$$|C_{z^{l-1}}[A^{l/s}(z)]| \leq r\frac{(A(r))^{l/s}}{r^l} \leq r\Big(\Big(\frac{A(r)}{r^s}\Big)^{1/s}\Big)^l\ , \tag{4.14}$$

and this decays exponentially fast as $l \to \infty$. □

The convergence rate of the series that appear in (3.6)-(3.14) can be read off from the following result.

Lemma 4.3. *Assume that A satisfies the conditions of Sec. 1 with $\varepsilon > \delta > 0$ such that $A(z)$ is analytic in $|z| < 1+\varepsilon$ and $|A(z)| < |z|^s$ in $1 < |z| < 1+\delta$ (no assumption on R). Let h be analytic in $|z| < 1+\varepsilon$. Then for any $r \in (1,1+\delta)$ we have*

$$|C_{z^j}[A^l(z)\,h(z)]| \leq \Big(\frac{A(r)}{r^s}\Big)^l \frac{M}{r^{j-ls}}\ , \qquad l = 1,2,...,j \geq ls\ , \tag{4.15}$$

where $M = \max\{|h(z)| \mid |z| = r\}$.

Proof. This follows, in a similar fashion as Lemma 4.2, from Cauchy's theorem and $A'(1) < s$. □

Example 4.4. Consider the Poisson case $A(z) = \exp(\lambda(z-1))$ with $0 \le \lambda < s$. We have pictured in Fig. 1 the set $\mathcal{S}_\vartheta$ in (4.4) for a number of values of $\vartheta := \lambda/s$ (although not permitted, $\vartheta = 1$ is included). The dots on the curves indicate the roots z_k for the case $s = 20$; this will be discussed at the end of this section. We compute

$$c_l = \frac{1}{l}\, C_{z^{l-1}}[A^{l/s}(z)] = \frac{1}{l}\, C_{z^{l-1}}[e^{\vartheta l(z-1)}] = e^{-l\vartheta}\, \frac{(l\vartheta)^{l-1}}{l!} \tag{4.16}$$

for $l = 1, 2, \ldots$. Hence $\mathcal{S}_\vartheta$ has the parametric representation

$$z_\vartheta(\alpha) = \sum_{l=1}^{\infty} e^{-l\vartheta}\, \frac{(l\vartheta)^{l-1}}{l!}\, e^{il\alpha} \;, \qquad \alpha \in [0, 2\pi] \;. \tag{4.17}$$

We observe that c_l is accurately approximated, using Stirling's formula, by

$$c_l \approx \frac{(\vartheta\, e^{1-\vartheta})^l}{\vartheta l\, \sqrt{2\pi l}} \;, \qquad l = 1, 2, \ldots \;, \tag{4.18}$$

where we note that $\vartheta\, e^{1-\vartheta}$ increases from 0 to 1 as ϑ increases from 0 to 1. Hence, even for $\vartheta = 1$ the representation in (4.17) makes sense.

Example 4.5. Consider the binomial case $A(z) = (p + qz)^n$ where $p,\ q \ge 0$, $p + q = 1$ and $A'(1) = nq < s$. We compute in this case

$$\begin{aligned} c_l &= \frac{1}{l}\, C_{z^{l-1}}[A^{l/s}(z)] = \frac{1}{l}\, C_{z^{l-1}}[(p+qz)^{nl/s}] \\ &= \frac{1}{l}\, p^{\frac{nl}{s}-l+1}\, q^{l-1} \binom{nl/s}{l-1} \;, \qquad l = 1, 2, \ldots \;, \end{aligned} \tag{4.19}$$

where we have used the notation

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdot \ldots \cdot(\alpha-k+1)}{k!} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\,\Gamma(k+1)} \;. \tag{4.20}$$

Let $\beta := n/s$. When $\beta = 1$ we have

$$c_l = p\, q^{l-1} \;, \qquad l = 1, 2, \ldots \;, \tag{4.21}$$

and there is exponential decay (when $\beta = 1$ we have $q < s/n = 1$). When $\beta > 1$, one has by Stirling's formula for $\Gamma(x+1)$,

$$c_l \approx \frac{p}{q}\, \frac{1}{\beta-1}\, \frac{1}{l\,\sqrt{2\pi l}}\, p^{l(\beta-1)}\, q^l \Big(\frac{\beta}{\beta-1}\Big)^{1/2} \Big[\frac{\beta^\beta}{(\beta-1)^{\beta-1}}\Big]^l \;, \tag{4.22}$$

whence there is exponential decay when

$$\frac{\beta^\beta}{(\beta-1)^{\beta-1}}\, p^{\beta-1}\, q < 1 \;. \tag{4.23}$$

For fixed p, q, the quantity at the left-hand side of (4.23) is maximal as a function of β at $\beta = 1/q$, with the value 1. Hence, since $\beta = n/s < 1/q$, we have exponential decay. Finally, when $0 < \beta < 1$, one has again by Stirling's formula and the formula $\Gamma(x)\,\Gamma(1-x) = \pi/\sin \pi x$,

$$c_l \approx \frac{p}{q}\, \frac{1}{(1-\beta)^2}\, \frac{(-1)^l \sin \pi l\beta}{l\,\sqrt{\frac{1}{2}\pi l}}\, p^{l(\beta-1)}\, q^l ((1-\beta)\beta)^{1/2} ((1-\beta)^{1-\beta}\, \beta^\beta)^l \;, \tag{4.24}$$

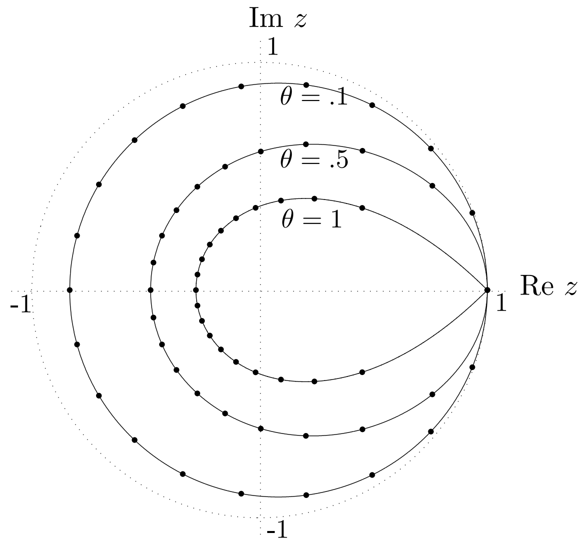

Figure 1: $\mathcal{S}_\vartheta$ for Poisson case, $\vartheta = .1, .5, 1$. The dots indicate $z_0, \ldots, z_{19}$ for $s = 20$.

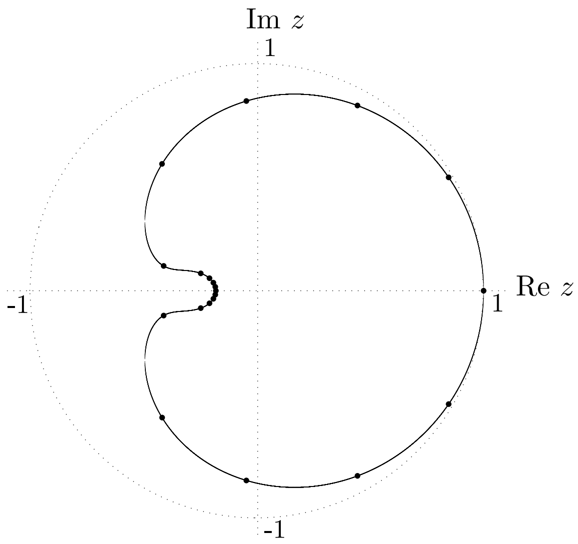

Figure 2: $\mathcal{S}_{A,s=2n}$ for binomial case, $q = .82$. The dots indicate $z_0, \ldots, z_{19}$ for $s = 20$.

whence there is exponential decay when

$$p^{\beta-1}\, q(1-\beta)^{1-\beta}\, \beta^{\beta} < 1 \ . \tag{4.25}$$

Note that the left-hand side of (4.25) increases from 0 to ∞ when q increases from 0 to 1 ($p = 1 - q$). In the critical case, where we have $=$ instead of $<$ in (4.25), there is still a $l^{-3/2}$-decay of the c_l. This critical case also arises in the following way. With $\beta = n/s$ we consider the equation

$$|p + qz|^{\beta} = |z| \tag{4.26}$$

for negative $z = -r \in [-1, 0)$. When $0 < \beta < 1$ and $p/q < 1$ this equation has at least one and at most three roots $z \in [-1, 0]$. The critical case now occurs when (4.26) has three roots of which two of them coincide.

In Figs. 2-4 we consider the case that $\beta = \frac{1}{2}$ and $s = 20$. The critical case now occurs for $q_0 = 2(\sqrt{2} - 1) = 0.828427125$. We have plotted the set

$$\mathcal{S}_{A,s=2n} = \{z \mid |z| \leq 1,\ |p + qz|^{1/2} = |z|\} \tag{4.27}$$

for $q = 0.82, 2(\sqrt{2} - 1), 0.83$. We observe that $\mathcal{S}_{A,s}$ turns from a smooth Jordan curve containing 0 (Fig. 2) into two separate closed curves when q passes q_0 (Fig. 4).

Lemma 4.6. *Assume that A satisfies the assumptions of Sec. 1 and that the radius of convergence, R, of the series in (4.5) > 1. Also assume that g is analytic in an open neighbourhood of $\{z_0(w) \mid |w| \leq 1\}$. Then $c_l(g) = l^{-1}\, C_{z^{l-1}}[A^{l/s}(z)\, g'(z)]$ has exponential decay, and there is an $R_g > 1$ such that*

$$g(z_0(w)) = g(0) + \sum_{l=1}^{\infty} c_l(g)\, w^l \ , \qquad |w| < R_g \ , \tag{4.28}$$

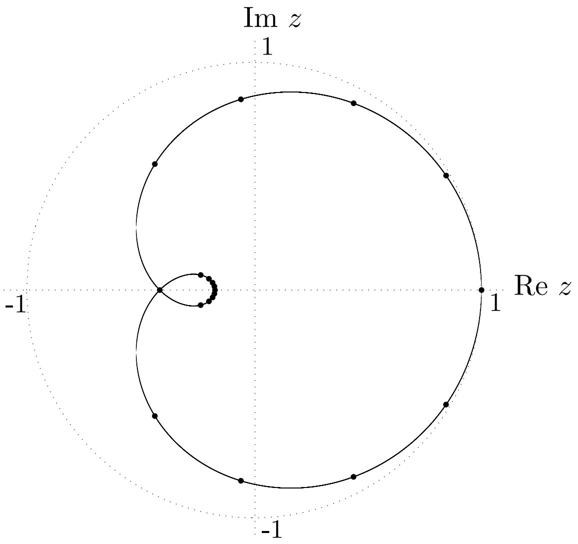

Figure 3: $\mathcal{S}_{A,s=2n}$ for binomial case, $q = 2(\sqrt{2}-1)$. The dots indicate $z_0, \ldots, z_{19}$ for $s = 20$.

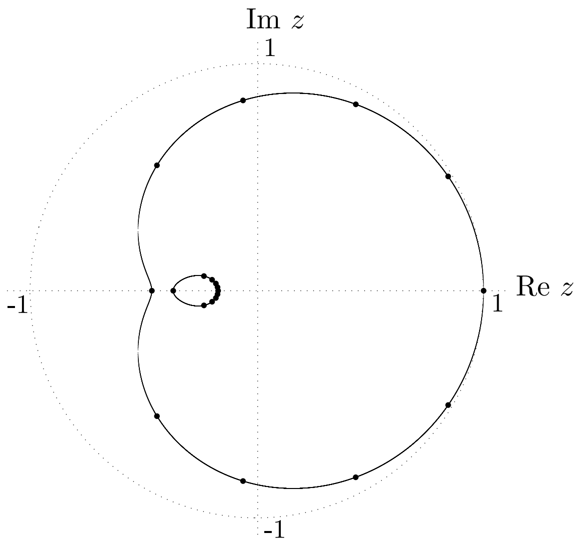

Figure 4: $\mathcal{S}_{A,s=2n}$ for binomial case, $q = .83$. The dots indicate $z_0, \ldots, z_{19}$ for $s = 20$.

with absolute convergence at the right-hand side of (4.28). In particular, we have

$$g(z(\alpha)) = g(0) + \sum_{l=1}^{\infty} c_l(g)\, e^{il\alpha} \;, \qquad \alpha \in [0, 2\pi] \;, \tag{4.29}$$

with absolute convergence at the right-hand side of (4.29).

Proof. There is an R_1, $1 < R_1 < R$, such that g is analytic in $\{z_0(w) \mid |w| < R_1\}$. With $1 < R_2 < R_1$ and $C_2 = \{z_0(R_2\, e^{i\alpha}) \mid \alpha \in [0, 2\pi]\}$, a Jordan curve with 0 in its interior, we have by Cauchy's theorem for $l = 1, 2, \ldots$

$$C_{z^{l-1}}[A^{l/s}(z)\, g'(z)] = \frac{1}{2\pi i} \int\limits_{C_2} \frac{A^{l/s}(z)\, g'(z)}{z^l}\, dz \;. \tag{4.30}$$

On C_2 we have $|A(z)/z^s| = R_2^{-s}$, whence

$$|C_{z^{l-1}}[A^{l/s}(z)\, g'(z)]| \le M\, R_2^{-l} \;, \qquad l = 1, 2, \ldots \;, \tag{4.31}$$

where $M = \max\{|g'(z)| \mid z \in C_2\}$. This shows exponential decay of $c_l(g)$. From this (4.28) easily follows with $R_g = R_2$ since $g(z_0(w)) = g(0) + \sum_{l=1}^{\infty} c_l(g)\, w^l$ holds in a neighbourhood of 0 by Lagrange's theorem. Finally, (4.29) is a direct consequence of (4.28). $\square$

Note. As one sees from the proof of Lemma 4.6 a geometric reformulation of the condition $R > 1$ reads: there is a Jordan curve J with 0 in its interior such that $A(z)$ is zero-free on and inside J while $|A(z)| < |z|^s$ on J.

We now make some comments on equidistant sampling of functions $g(z(\alpha))$ under the conditions of Lemma 4.6. The zeros $z_k = z(2\pi k/s)$, $k = 0, 1, \ldots, s-1$, can be computed from

(4.29) by taking $g(z)=z$ and $\alpha=2\pi k/s$. Hence

$$z_k=z(2\pi k/s)=\sum_{l=1}^{\infty}\frac{1}{l}\,C_{z^{l-1}}[A^{l/s}(z)]\,e^{2\pi ikl/s}\ , \qquad k=0,1,...,s-1\ . \tag{4.32}$$

As mentioned earlier, the z_k are displayed as dots for

(a) $A(z)=\exp(\lambda(z-1))$, $s=20$, $\lambda=2,10,20$,

(b) $A(z)=(p+qz)^n$, $s=20$, $n=10$, $q=0.82, 2(\sqrt{2}-1), 0.83$,

in Fig. 1 and Figs. 2-4, respectively. For all cases, except the last one in (b) we can use (4.32); for the latter case we had to use a different (numerical) procedure.

When the conditions of Lemma 4.6 are satisfied, we see immediately from (4.29) and

$$\sum_{k=0}^{s-1}e^{2\pi ikl/s}=\begin{cases} s\ , & l=0(\mathrm{mod}\,s)\ , \\ 0\ , & l\neq 0(\mathrm{mod}\,s)\ , \end{cases} \tag{4.33}$$

that

$$\sum_{k=0}^{s-1}g(z_k)=\sum_{k=0}^{s-1}g((z(2\pi k/s))=s\,g(0)+s\sum_{l=1}^{\infty}c_{ls}(g)\ . \tag{4.34}$$

5 Expressing $\sum(1-z_k)^{-1}$ and $\sum z_k(1-z_k)^{-2}$ in terms of aliasing series

In this section we express the series

$$\sum_{k=1}^{s-1}\frac{1}{1-z_k}\ , \qquad \sum_{k=1}^{s-1}\frac{z_k}{(1-z_k)^2}\ , \tag{5.1}$$

in terms of aliasing series under the conditions on A of Sec. 1 and the assumption that the series in (4.5) has radius of convergence $R>1$.

To apply Lemma 4.6, we need to regularize the functions

$$g_1(z)=\frac{1}{1-z}\ , \qquad g_2(z)=\frac{z}{(1-z)^2}=\frac{1}{(1-z)^2}-\frac{1}{1-z}\ , \tag{5.2}$$

at $z=1$. This must be done in a clever way so that the regularized functions are manageable. We consider g_1 and g_2 on and around the Szegö curve $\mathcal{S}_{A,s}=\{z_0(w)\mid w=e^{i\alpha},\alpha\in[0,2\pi)\}$, an so it would be pleasant when the regularizing functions assume a simple form on $\mathcal{S}_{A,s}$. We thus propose to subtract from g_1 and g_2 in (5.2) the functions

$$h_1(z)=\frac{B}{1-z\,A^{-1/s}(z)}\ , \qquad h_2(z)=\frac{C}{(1-z\,A^{-1/s}(z))^2}+\frac{D}{1-z\,A^{-1/s}(z)}\ , \tag{5.3}$$

for which we have

$$h_1(z_0(w))=\frac{B}{1-w}\ , \qquad h_2(z_0(w))=\frac{C}{(1-w)^2}+\frac{D}{1-w}\ , \tag{5.4}$$

on account of (4.8). In (5.2) we must choose B and C, D such that $g_1 - h_1$ and $g_2 - h_2$ are regular at $z = 1$; this can be done indeed since $1 - z\,A^{-1/s}(z)$ has a first order zero at $z = 1$ (as $A'(1) < s$). Noting that $z_k = z_0(\exp(2\pi ik/s))$, we have from (5.3) that $\sum_{k=1}^{s-1} h_i(z_k)$ are computationally manageable. In fact one has explicitly

$$\sum_{k=1}^{s-1} \frac{1}{1 - e^{2\pi ik/s}} = \tfrac{1}{2}(s-1)\ , \qquad \sum_{k=1}^{s-1} \frac{1}{(1 - e^{2\pi ik/s})^2} = -\tfrac{1}{12}(s-1)(s-5)\ . \tag{5.5}$$

The decisive reason to choose h_1, h_2 of the above type is the following result that shows that subtraction of h_i does not lead to unmanageable expressions in the aliasing series.

Lemma 5.1. *Let f be analytic in a neighbourhood of 0. Then*

$$\frac{1}{l}\, C_{z^{l-1}}\Big[A^{l/s}(z)\,\frac{d}{dz}\,(f(z\,A^{-1/s}(z)))\Big] = C_{w^l}[f(w)]\ . \tag{5.6}$$

Proof. We have by Lagrange's theorem, see the beginning of Sec. 2,

$$\begin{aligned}
&\frac{1}{l}\, C_{z^{l-1}}\Big[A^{l/s}(z)\,\frac{d}{dz}\,(f(z\,A^{-1/s}(z)))\Big] \\
&= \frac{1}{l!}\Big(\frac{d}{dz}\Big)^{l-1}\Big[A^{l/s}(z)\,\frac{d}{dz}\,(f(z\,A^{-1/s}(z)))\Big]\,(z=0) \\
&= C_{w^l}[f(z\,A^{-1/s}(z)) \text{ where } z \text{ satisfies } z\,A^{-1/s}(z) = w] \\
&= C_{w^l}[f(w)]\ ,
\end{aligned} \tag{5.7}$$

as required. □

We finally consider the issue of choosing B and C, D properly in (5.3). Thus we let $g_i^R := g_i - h_i$, $i = 1, 2$. A lengthy but otherwise elementary computation shows that for g_1^R we need to take

$$B = 1 - s^{-1}\,A'(1) \tag{5.8}$$

so that g_1^R is indeed regular at $z = 1$, with value

$$\begin{aligned}
g_1^R(1) &= \frac{1}{1-z} - \frac{B}{1 - z\,A^{-1/s}(z)}\Big|_{z=1} \\
&= \frac{s^{-1}\,A'(1) - \frac{1}{2}[s^{-1}(s^{-1}+1)(A'(1))^2 - s^{-1}\,A''(1)]}{1 - s^{-1}\,A'(1)}
\end{aligned} \tag{5.9}$$

at $z = 1$. For the regularization of g_2 we need to take

$$C = (1 + a^{[1]})^2\ , \qquad D = -1 - 3a^{[1]} - a^{[2]}\ , \tag{5.10}$$

and then

$$\begin{aligned}
g_2^R(1) &= \frac{z}{(1-z)^2} - \frac{C}{(1 - z\,A^{-1/s}(z))^2} - \frac{D}{1 - z\,A^{-1/s}(z)}\Big|_{z=1} \\
&= \frac{a^{[2]} + \frac{1}{3}a^{[3]}}{1 + a^{[1]}} + \frac{a^{[1]} + \frac{1}{2}a^{[2]}}{1 + a^{[1]}}\Big(1 - \frac{a^{[1]} + \frac{1}{2}a^{[2]}}{1 + a^{[1]}}\Big)\ ,
\end{aligned} \tag{5.11}$$

where

$$a^{[i]} = \left(\frac{d}{dz}\right)^i A^{-1/s}(z)\Big|_{z=1} \;, \qquad i = 1, 2, 3 \;. \tag{5.12}$$

We are now ready to prove Thms. 3.1-3.2 in Sec. 3. We have by Lemma 4.6 with $g = g_1^R$ and (5.5) that

$$\begin{aligned} \sum_{k=1}^{s-1} \frac{1}{1-z_k} &= \sum_{k=1}^{s-1} \frac{B}{1-e^{2\pi i k/s}} + \sum_{k=1}^{s-1} g_1^R(z_k) \\ &= \tfrac{1}{2}\, B(s-1) - g_1^R(1) + \sum_{k=0}^{s-1} g_1^R(z_k) \\ &= \tfrac{1}{2}\, B(s-1) - g_1^R(1) + s\, g_1^R(0) + s \sum_{l=1}^{\infty} c_{ls}(g_1^R) \;. \end{aligned} \tag{5.13}$$

Furthermore,

$$\begin{aligned} c_{ls}(g_1^R) &= \frac{1}{ls}\, C_{z^{ls-1}}[A^l(z)(g_1^R)'(z)] \\ &= \frac{1}{ls}\, C_{z^{ls-1}}\Big[A^l(z)\,\frac{1}{(1-z)^2}\Big] - \frac{B}{ls}\, C_{z^{ls-1}}\Big[A^l(z)\Big(\frac{1}{1-s^{-1}\,A(z)}\Big)'\Big] \;. \end{aligned} \tag{5.14}$$

Using $(1-z)^{-2} = \sum_{j=0}^{\infty} (j+1)\, z^j$ and applying Lemma 5.1 we then get that

$$\begin{aligned} c_{ls}(g_1^R) &= \frac{1}{ls} \sum_{j=0}^{ls-1} (ls-j)\, C_{z^j}[A^l(z)] - C_{w^{ls}}\Big[\frac{B}{1-w}\Big] \\ &= \frac{1}{ls} \sum_{j=0}^{ls-1} (ls-j)\, C_{z^j}[A^l(z)] - B \;. \end{aligned} \tag{5.15}$$

To bring the right-hand side of (5.15) in its final form, we observe that $c_{ls}(g_1^R) \to 0$ as $l \to \infty$ and that

$$1 = A^l(1) = \sum_{j=0}^{\infty} C_{z^j}[A^l(z)] \;, \tag{5.16}$$

$$l\, A'(1) = \frac{d}{dz}\,[A^l(z)]\,(z=1) = \sum_{j=0}^{\infty} j\, C_{z^j}[A^l(z)] \;. \tag{5.17}$$

This implies that $B = 1 - s^{-1}\, A'(1)$, which agrees with (5.8), and that

$$c_{ls}(g_1^R) = \frac{1}{ls} \sum_{j=ls}^{\infty} (j-ls)\, C_{z^j}[A^l(z)] \;. \tag{5.18}$$

Therefore we arrive at (noting that $g_1^R(0) = 1 - B$)

$$\sum_{k=1}^{s-1} \frac{1}{1-z_k} = (1 - \tfrac{1}{2}\,B)\,s - \tfrac{1}{2}\,B - g_1^R(1) + \sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=ls}^{\infty} C_{z^j}[A^l(z)] \ . \tag{5.19}$$

The proof of Thm. 3.1 is completed by a rather long but otherwise elementary computation, using the expressions in (5.8) and (5.9) for B and $g_1^R(1)$ and the fact that

$$A'(1) = \mu_A \ , \qquad A''(1) = \sigma_A^2 + \mu_A^2 - \mu_A \ . \tag{5.20}$$

The procedure for computation of $\sum_{k=1}^{s-1} z_k(1-z_k)^{-2}$ is entirely the same as the one for $\sum_{k=1}^{s-1}(1-z_k)^{-1}$, although quite a bit more elaborate. Accordingly, using both items in (5.5) and Lemma 4.6 with $g = g_2^R$ we get as in (5.19) that (using $g_2^R(0) = -C - D$)

$$\begin{aligned} \sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} &= \ - \ \tfrac{1}{12}\,C(s-1)(s-5) + \tfrac{1}{2}\,D(s-1) - g_2^R(1) - s(C+D) \\ &+ \ \ s\sum_{l+1}^{\infty} c_{ls}(g_2^R) \ . \end{aligned} \tag{5.21}$$

Using that $(z(1-z)^{-2})' = \sum_{j=0}^{\infty} j^2\, z^{j-1}$ we find in a similar fashion as in (5.15) from Lemma 5.1 that

$$\begin{aligned} c_{ls}(g_2^R) &= \frac{1}{ls}\,C_{z^{ls-1}}\Big[A^l(z)\Big(\frac{z}{(1-z)^2}\Big)'\Big] - C_{w^l}\Big[\frac{C}{(1-w)^2} + \frac{D}{1-w}\Big] \\ &= \frac{1}{ls}\sum_{j=0}^{ls-1}(ls-j)^2\,C_{z^j}[A^l(z)] - C(l+1) - D \ . \end{aligned} \tag{5.22}$$

To bring (5.22) in its final form, we observe that $c_{ls}(g_2^R) \to 0$ as $l \to \infty$ by Lemma 4.6, and we use (5.16) and (5.17) together with

$$\sum_{j=0}^{\infty} j^2\,C_{z^j}[A^l(z)] = l(l-1)(A'(1))^2 + l\,A''(1) + l\,A'(1) \tag{5.23}$$

and (5.20). This yields (in agreement with (5.10))

$$C + D = \frac{1}{s}\,\sigma_A^2 \ , \qquad C = \Big(1 - \frac{1}{s}\,\mu_A\Big)^2 \ , \tag{5.24}$$

and

$$c_{ls}(g_2^R) = \frac{-1}{ls}\sum_{j=ls}^{\infty}(j-ls)^2\,C_{z^j}[A^l(z)] \ . \tag{5.25}$$

We then find that

$$\begin{aligned} \sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} &= -\tfrac{1}{12}\,C(s-1)(s-5) + \tfrac{1}{2}\,D(s-1) - g_2^R(1) - s(C+D) \\ &- \sum_{l=1}^{\infty}\frac{1}{l}\sum_{j=ls}^{\infty}(j-ls)^2\,C_{z^j}[A^l(z)] \ . \end{aligned} \tag{5.26}$$

The proof of Thm. 3.2 is then completed by an extremely long but otherwise elementary calculation in which the two members in (3.5) are shown to be equal with C, D and $g_2^R(1)$ given through (5.10)–(5.12) and (5.24).

6 The stationary queue length distribution

In this section we derive explicit formulas for the x_j, $j = 0, ..., s$, and for

$$c = \sum_{j=0}^{s-1} x_j \ , \qquad d = \sum_{j=0}^{s} x_j \ . \tag{6.1}$$

We do this using our approach under the assumptions on A of Sec. 1 and the condition that the series in (4.5) has radius of convergence $R > 1$.

We denote

$$Q(z) := \sum_{j=0}^{s-1} (z^s - z^j)\, x_j =: \sum_{j=0}^{s} q_j\, z^j \tag{6.2}$$

so that (1.5) can be written in the form

$$A(z)\, X(z) = -A(z)\, Q(z) + z^s\, X(z) \ . \tag{6.3}$$

Then it follows that

$$x_j = -q_j \ , \quad j = 0, ..., s-1 \ ; \qquad x_s = -q_s - a_0^{-1} q_0 \ . \tag{6.4}$$

Since $X(z)$ is analytic in a disk $|z| < 1 + \varepsilon$ with $\varepsilon > 0$, the s^{th} degree polynomial Q cancels all s zeros of $z^s - A(z)$ in $|z| \leq 1$, see (1.5). Hence

$$Q(z) = \gamma \prod_{k=0}^{s-1} (z - z_k) \tag{6.5}$$

for some constant γ. Differentiating (6.3) and setting $z = 1$ yields $Q'(1) = s - \mu_A$ while from (6.5) noting that $z_0 = 1$ we get $Q'(1) = \gamma \prod_{k=1}^{s-1} (1 - z_k)$. Hence

$$\gamma = (s - \mu_A) \prod_{k=1}^{s-1} (1 - z_k)^{-1} \ . \tag{6.6}$$

Furthermore, from (6.5) and (6.2) we see that

$$\gamma = C_{z^s}[Q(z)] = \sum_{j=0}^{s-1} x_j = c \ , \tag{6.7}$$

with c given in (6.1). We thus have that

$$Q(v) = (-1)^s\, c \prod_{k=1}^{s-1} z_k \prod_{k=0}^{s-1} \left(1 - \frac{v}{z_k}\right) \ , \tag{6.8}$$

and then (6.4) shows that is enough to find explicit formulas for

$$c = (s - \mu_A) \prod_{k=1}^{s-1} (1 - z_k)^{-1} \ , \quad \prod_{k=0}^{s-1} z_k \ , \quad C_{v^j}\Big[\prod_{k=0}^{s-1} \left(1 - \frac{v}{z_k}\right)\Big] \ , \qquad j = 1, ..., s-1 \ . \tag{6.9}$$

We start by considering $\prod_{k=1}^{s-1}(1-z_k)^{-1}$ and to that end we regularize $g_3(z)=\ln(1-z)$ at $z=1$ by setting

$$g_3^R(z)=\ln(1-z)-\ln(1-z\,A^{-1/s}(z))\ . \tag{6.10}$$

Then g_3^R is analytic in an open neighbourhood of $\{z\mid |z|^s\leq A(z)\}$, and

$$g_3^R(1)=-\ln(1-\mu_A/s)\ ,\qquad g_3^R(0)=0\ . \tag{6.11}$$

Also, we have $z_k\,A^{-1/s}(z_k)=\exp(2\pi ik/s)$, $k=0,1,...,s-1$, and there is the identity

$$\sum_{k=1}^{s-1}\ln(1-e^{2\pi ik/s})=\ln s\ . \tag{6.12}$$

We thus obtain as before from the above that

$$\begin{aligned}\sum_{k=1}^{s-1}\ln(1-z_k) &= \sum_{k=1}^{s-1}\ln(1-e^{2\pi ik/s})+\sum_{k=1}^{s-1}g_3^R(z(2\pi k/s))\\ &= \ln(s-\mu_A)+s\sum_{l=1}^{\infty}c_{ls}(g_3^R)\ .\end{aligned} \tag{6.13}$$

Here we have, also as before, from Lemma 4.6 and Lemma 5.1

$$\begin{aligned}c_{ls}(g_3^R) &= \frac{1}{ls}\,C_{z^{ls-1}}[A^l(z)(\ln(1-z))']-C_{w^{ls}}[\ln(1-w)]\\ &= \frac{1}{ls}\sum_{j=ls}^{\infty}C_{z^j}[A^l(z)]\ .\end{aligned} \tag{6.14}$$

Hence we get

$$\sum_{k=1}^{s-1}\ln(1-z_k)=\ln(s-\mu_A)+\sum_{l=1}^{\infty}\frac{1}{l}\sum_{j=ls}^{\infty}C_{z^j}[A^l(z)]\ , \tag{6.15}$$

so that

$$c=\exp\Big\{-\sum_{l=1}^{\infty}\frac{1}{l}\sum_{j=ls}^{\infty}C_{z^j}[A^l(z)]\Big\}\ . \tag{6.16}$$

We next compute $\prod_{k=0}^{s-1}z_k$. To that end we note that

$$z_k=e^{2\pi ik/s}\,A^{1/s}(z(2\pi k/s))\ , \tag{6.17}$$

so that

$$\prod_{k=0}^{s-1}z_k=(-1)^{s-1}\,\exp\Big\{\sum_{k=0}^{s-1}\ln\Big[A^{1/s}(z(2\pi k/s))\Big]\Big\}\ . \tag{6.18}$$

The function $g_4(z):=\ln\,[A^{1/s}(z)]$ is analytic in a neighbourhood of $\{z\mid |z|^s\leq|A(z)|\}$, and we have

$$g_4(1)=0\ ,\qquad g_4(0)=\frac{1}{s}\ln a_0\ . \tag{6.19}$$

Hence using our approach there follows

$$\sum_{k=0}^{s-1} \ln \Big[A^{1/s}(z(2\pi k/s))\Big] = \ln a_0 + s \sum_{l=1}^{\infty} c_{ls}(g_4) \ . \tag{6.20}$$

The $c_{ls}(g_4)$ follow from

$$\begin{aligned} c_{ls}(g_4) &= \frac{1}{(ls)!}\left(\frac{d}{dz}\right)^{ls-1} [A^l(z)(\ln[A^{1/s}(z)])'](z=0) \\ &= \frac{1}{s(ls)!}\left(\frac{d}{dz}\right)^{ls-1} [A^{l-1}(z)\,A'(z)]\,(z=0) = \frac{1}{ls}\,C_{z^{ls}}[A^l(z)] \ . \end{aligned} \tag{6.21}$$

It thus follows that

$$\prod_{k=1}^{s-1} z_k = (-1)^{s-1}\,a_0\,\exp\Big\{\sum_{l=1}^{\infty}\frac{1}{l}\,C_{z^{ls}}[A^l(z)]\Big\} \ . \tag{6.22}$$

We then find from (6.4) and (6.8) that

$$x_0 = -q_0 = (-1)^s\,c\prod_{k=1}^{s-1} z_k = a_0\,\exp\Big\{-\sum_{l=1}^{\infty}\frac{1}{l}\sum_{j=ls+1}^{\infty} C_{z^j}[A^l(z)]\Big\} \ . \tag{6.23}$$

Moreover, from $Q(1) = 0$ and (6.3) we have

$$d = \sum_{j=0}^{s} x_j = a_0^{-1}\,x_0 = \exp\Big\{-\sum_{l=1}^{\infty}\frac{1}{l}\sum_{j=ls+1}^{\infty} C_{z^j}[A^l(z)]\Big\} \ . \tag{6.24}$$

We conclude by computing the x_i, $i = 1, ..., s-1$. Note that

$$x_i = x_0\,C_{v^i}\Big[\prod_{k=0}^{s-1}\Big(1 - \frac{v}{z_k}\Big)\Big] \ , \qquad i = 1, ..., s-1 \ . \tag{6.25}$$

We shall consider, using (6.17) and the Taylor expansion of $\ln(1-x)$ around $x = 0$, the expression

$$\sum_{k=0}^{s-1}\ln\Big(1 - \frac{v}{z_k}\Big) = -\sum_{j=1}^{\infty}\frac{v^j}{j}\sum_{k=0}^{s-1} A^{-j/s}(z(2\pi k/s))\,e^{-2\pi ijk/s} \ . \tag{6.26}$$

The x_i in (6.25) are completely determined by the terms at the right-hand side of (6.26) with $j = 1, ..., s-1$. Thus we consider for $j = 1, ..., s-1$ the 2π-periodic functions

$$A^{-j/s}(z(\alpha)) = A^{-j/s}(0) + \sum_{l=1}^{\infty} c_l[A^{-j/s}]\,e^{il\alpha} \ . \tag{6.27}$$

The $c_l[A^{-j/s}]$ are given here as

$$\begin{aligned} c_l[A^{-j/s}] &= \frac{1}{l!}\left(\frac{d}{dz}\right)^{l-1} [A^{l/s}(z)(A^{-j/s})'(z)]\,(z=0) \\ &= \frac{-1}{l!}\,\frac{j}{s}\left(\frac{d}{dz}\right)^{l-1} [A^{-1-(l-j)/s}(z)\,A'(z)]\,(z=0) \ . \end{aligned} \tag{6.28}$$

It is seen from (6.28) that

$$c_j[A^{-j/s}] = \frac{-j}{s}\, C_{z^j}\,[\ln\,[A(z)]]\ , \tag{6.29}$$

$$c_l[A^{-j/s}] = \frac{-j}{l-j}\, C_{z^l}\,[A^{(l-j)/s}(z)]\ , \qquad l \neq j\ . \tag{6.30}$$

Since $A(0) = a_0$, we thus get that

$$A^{-j/s}(z(\alpha))\, e^{-ij\alpha} = \; a_0^{-j/s}\, e^{-ij\alpha} - \frac{j}{s}\, C_{z^j}\,[\ln A(z)] - j \sum_{\substack{l=-j+1,\\ l\neq 0}}^{\infty} \frac{1}{l}\, C_{z^{l+j}}\,[A^{l/s}(z)]\, e^{il\alpha}\ . \tag{6.31}$$

Therefore, for $j = 1, ..., s-1$ by sampling theory,

$$\sum_{k=0}^{s-1} A^{-j/s}(z(2\pi k/s))\, e^{-2\pi ijk/s} = \; -j\, C_{z^j}\,[\ln\,[A(z)]] - \sum_{l=1}^{\infty} \frac{j}{l}\, C_{z^{ls+j}}\,[A^l(z)]\ . \tag{6.32}$$

This gives, see (6.25)–(6.26), for $i = 1, ..., s-1$ that

$$x_i = x_0\, C_{v^i}\Big[\exp\Big\{\sum_{j=1}^{s-1} v^j \Big(C_{z^j}\,[\ln\,[A(z)]] + \sum_{l=1}^{\infty} \frac{1}{l}\, C_{z^{ls+j}}\,[A^l(z)]\Big)\Big\}\Big]\ . \tag{6.33}$$

Since we consider $i = 1, ..., s-1$ in (6.33) the summation over j may be extended to all $j = 1, 2, \ldots$. Noting that

$$\sum_{j=1}^{\infty} v^j\, C_{z^j}\,[\ln\,[A(z)]] = \ln A(v) - \ln a_0\ , \tag{6.34}$$

and that $x_0 = a_0\, d$, see (6.24), we arrive for $i = 1, ..., s-1$ at

$$x_i = d\, C_{v^i}\Big[A(v)\, \exp\Big\{\sum_{j=1}^{s-1} v^j \sum_{l=1}^{\infty} \frac{1}{l}\, C_{z^{ls+j}}\,[A^l(z)]\Big\}\Big]\ . \tag{6.35}$$

We finally have, see (6.1), (6.16) and (6.24), that

$$x_s = d - c = d\Big(1 - \exp\Big\{-\sum_{l=1}^{\infty} \frac{1}{l}\, C_{z^{ls}}\,[A^l(z)]\Big\}\Big)\ . \tag{6.36}$$

We thus have computed c, d and x_j, $j = 0, ..., s$. It is an immediate consequence of (1.5) and the definition of c as $\sum_{j=0}^{s-1} x_j$ that

$$X(z) - c\, A(z) = A(z) \sum_{j=0}^{\infty} x_{j+s}\, z^j\ . \tag{6.37}$$

This implies that $x_{s+1}, x_{s+2}, ...$ can be computed recursively from $x_0, ..., x_s$ and c.

7 Spitzer's identity and Wiener-Hopf factorization

In this section we show how the results in Sec. 3 can be alternatively derived from Spitzer's identity given by (3.9). Since the methods that are used stem from the fields of applied probability and stochastic processes, parts of the derivation are sketched, where we do give references to other places for a more complete mathematical underpinning.

Because of the discrete nature of the queue under investigation we make the change of variables $e^{-u} \to z$ in (3.9) yielding

$$E(z^W) = \exp\Big\{\sum_{l=1}^{\infty} \frac{1}{l}\, E(z^{\max(S_l,0)} - 1)\Big\}, \quad |z| \leq 1 , \tag{7.1}$$

where, as before, $S_l = \sum_{i=1}^{l} (A_i - s)$. We first show how expression (7.1) can be derived, analogously to [4] p. 338 for the continuous-time case, using Wiener-Hopf factorization (see [3]). From recursion (3.8) we have

$$\begin{aligned} E(z^{W_{t+1}}) &= P(W_t \leq s - A_{t-1}) + E(z^{W_t + A_{t-1} - s}\mathbf{1}\{W_t > s - A_{t-1}\}) \\ &= P(W_t \leq s - A_{t-1}) + E(z^{W_t + A_{t-1} - s}) - E(z^{W_t + A_{t-1} - s}\mathbf{1}\{W_t \leq s - A_{t-1}\}) , \end{aligned} \tag{7.2}$$

where $\mathbf{1}\{B\} = 1$ if B holds and 0 otherwise. Letting $t \to \infty$ and observing that W_t and A_{t-1} are independent then yields

$$E(z^W)(1 - z^{-s}A(z)) = P(W \leq s - A) - E(z^{W + A - s}\mathbf{1}\{W \leq s - A\}) . \tag{7.3}$$

We denote the right-hand side of (7.3) as $-W_-(z)$ and $E(z^W)$ as $W_+(z)$, which gives

$$W_+(z)(1 - z^{-s}A(z)) = -W_-(z) . \tag{7.4}$$

This basic identity is the starting point for the remaining analysis, for which we proceed in two ways: (i) the general way using no knowledge on the zeros of $1 - z^{-s}A(z)$, and (ii) the queue-specific way using an explicit factorization of $1 - z^{-s}A(z)$.

(i) Using

$$\frac{1}{1-z} = \exp\{-\ln(1-z)\} = \exp\Big\{\sum_{l=1}^{\infty} \frac{z^l}{l}\Big\}, \quad |z| < 1 , \tag{7.5}$$

we have (with $S_l = \sum_{i=1}^{l} (A_i - s)$) that

$$\begin{aligned} (1 - z^{-s}A(z))^{-1} &= \exp\Big\{\sum_{l=1}^{\infty} \frac{1}{l}(z^{-s}A(z))^l\Big\} \\ &= \exp\Big\{\sum_{l=1}^{\infty} \frac{1}{l}E(z^{S_l}\mathbf{1}\{S_l > 0\})\Big\} \cdot \exp\Big\{\sum_{l=1}^{\infty} \frac{1}{l}E(z^{S_l}\mathbf{1}\{S_l \leq 0\})\Big\}. \end{aligned} \tag{7.6}$$

Substituting (7.6) into (7.4) yields

$$W_+(z)\exp\Big\{-\sum_{l=1}^{\infty} \frac{1}{l}E(z^{S_l}\mathbf{1}\{S_l > 0\})\Big\} = -W_-(z)\exp\Big\{\sum_{l=1}^{\infty} \frac{1}{l}E(z^{S_l}\mathbf{1}\{S_l \leq 0\})\Big\} . \tag{7.7}$$

The left-hand side and right-hand side of (7.7) are analytic in $|z| < 1$ and $|z| > 1$, respectively, and continuous up to $|z| = 1$. Moreover, the left-hand side and right-hand side of (7.7) are bounded (see [4] p.338, [9] p.287) and analytic in $|z| < 1$ and $|z| > 1$, respectively. Therefore, their analytic continuation contains no singularities in the entire complex plane, whence upon using Liouville's theorem (see e.g. [4]) the left-hand side of (7.7) is constant, i.e.

$$W_+(z) = K \exp\Big\{ \sum_{l=1}^{\infty} \frac{1}{l} E(z^{S_l} \mathbf{1}\{S_l > 0\}) \Big\} . \tag{7.8}$$

The constant K follows from $W_+(1) = 1$ yielding

$$K = \exp\Big\{ - \sum_{l=1}^{\infty} \frac{1}{l} P(S_l > 0) \Big\} . \tag{7.9}$$

Upon inspection, one sees that the right-hand sides of (7.1) and (7.8) are the same.

(ii) An alternative way to construct a decomposition, starting from (7.4), is to invoke the following explicit factorization

$$1 - z^{-s} A(z) = \frac{z^s - A(z)}{z^s} = \frac{z^s - A(z)}{\prod_{k=0}^{s-1}(z - z_k)} \cdot \frac{\prod_{k=0}^{s-1}(z - z_k)}{z^s} , \tag{7.10}$$

where the first and second factor on the right-hand side of (7.10) are analytic and bounded in $|z| \leq 1$ and $|z| \geq 1$, respectively. Substituting (7.10) into (7.4) gives

$$W_+(z) \frac{z^s - A(z)}{\prod_{k=0}^{s-1}(z - z_k)} = -W_-(z) \frac{z^s}{\prod_{k=0}^{s-1}(z - z_k)} . \tag{7.11}$$

From Liouville's theorem it then follows that

$$W_+(z) = K \frac{(z-1)\prod_{k=1}^{s-1}(z - z_k)}{z^s - A(z)} , \tag{7.12}$$

where K again follows from $W_+(1) = 1$, i.e.

$$K^{-1} = \lim_{z \to 1} \frac{(z-1)\prod_{k=1}^{s-1}(z - z_k)}{z^s - A(z)} = \frac{\prod_{k=1}^{s-1}(1 - z_k)}{s - \mu_A}. \tag{7.13}$$

So we have for $W_+(z)$ the two expressions given by (7.8) and (7.12), respectively, and since by definition $X(z) = A(z) W_+(z)$, we have for $X(z)$ the expressions

$$\begin{aligned} X(z) &= A(z) \exp\Big\{ - \sum_{l=1}^{\infty} \frac{1}{l} P(S_l > 0) \Big\} \exp\Big\{ \sum_{l=1}^{\infty} \frac{1}{l} E(z^{S_l} \mathbf{1}\{S_l > 0\}) \Big\} && (7.14) \\ &= \frac{A(z)(z-1)(s - \mu_A)}{z^s - A(z)} \prod_{k=1}^{s-1} \frac{z - z_k}{1 - z_k} . && (7.15) \end{aligned}$$

Note that (7.15) also follows from substituting (6.5) into (6.3).

The mean and variance of X follow from $\mu_X = X'(1)$ and $\sigma_X^2 = X''(1) + X'(1) - X'(1)^2$. Then, differentiating (7.15) results in expressions (1.9) and (1.10), while differentiating (7.14) gives expressions (3.6) and (3.7). Moreover, we have shown that (3.6) and (3.7) can be derived from Fourier sampling on generalized Szegö curves.

Next, observe that

$$\exp\Big\{-\sum_{l=0}^{\infty}\frac{1}{l}P(S_l>0)\Big\}=\exp\Big\{-\sum_{l=1}^{\infty}\frac{1}{l}\sum_{j=ls+1}^{\infty}C_{z^j}[A^l(z)]\Big\}=d\ , \tag{7.16}$$

as in (6.24), and that the expression (3.12) for the x_i, $i=0,1,\ldots,s-1$ also follows from (7.14). It even follows that

$$x_i=d\,C_{v^i}\Big[A(v)\,\exp\Big\{\sum_{j=1}^{\infty}v^j\sum_{l=1}^{\infty}\frac{1}{l}\,C_{z^{ls+j}}[A^l(z)]\Big\}\Big]\ ,\quad i=0,1,\ldots, \tag{7.17}$$

and thus holding for all $\{x_i\}$.

From a numerical viewpoint, we might say that one can follow two courses in dealing with the queue under investigation: either determine the $s-1$ zeros of $A(z)=z^s$ within the unit disk, or calculate the infinite sum of power series coefficients of $A^l(z)$, $l=0,1,\ldots$, around $z=0$, up to a certain level. A comparison of these two alternatives is drawn in [8].

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