Analytic computation schemes for the discrete-time bulk service queue

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Abstract

In commonly used root-finding approaches for the discrete-time bulk service queue, the stationary queue length distribution follows from the roots inside or outside the unit circle of a characteristic equation. We present analytic representations of these roots in the form of sample values of periodic functions with analytically given Fourier series coefficients, making these approaches more transparent and explicit. The resulting computational scheme is easy to implement and numerically stable. We also discuss a method to determine the roots by applying successive substitutions to a fixed point equation. We outline under which conditions this method works, and compare these conditions with those needed for the Fourier series representation. Finally, we present a solution for the stationary queue length distribution that does not depend on roots. This solution is explicit and well-suited for determining tail probabilities up to a high accuracy, as demonstrated by some numerical examples.

keywords: discrete-time bulk service, multi-server, roots, stationary distribution, Szegö curve, Spitzer's identity.

1 Introduction and motivation

During the last two decades, discrete-time queueing models have been applied to model digital communication systems such as multiplexers and packet switches. In this field, the multi-server or bulk service queue fulfills a key role due to its wide range of applications, among which ATM switching elements [3], data transmission over satellites [29], high performance serial busses [22], and cable access networks [12].

The discrete-time bulk service queue is defined by the recursion

$$X_{n+1} = \max\{X_n - s, 0\} + A_n. \tag{1}$$

Here, time is assumed to be slotted, X_n denotes the queue length at the beginning of slot n, A_n denotes the number of new arriving customers during slot n, and s denotes the fixed number of customers that can be served during one slot. The sequence of A_n is assumed to be independent and identically distributed (i.i.d.) according to a discrete random variable A with probability generating function (pgf) $A(z)$. Without loss of generality we assume throughout that $P(A = 0) > 0$. The pgf of the stationary queue length in the discrete-time bulk service queue was first derived by Bruneel & Wuyts [4], although the same pgf occurs in earlier work on the $D/G/1$ queue by Servi [25] and on bulk queues by Powell [24]. The solution requires finding the roots of $z^s = A(z)$ within the unit circle. Zhao & Campbell [30] presented a full solution for the stationary queue length distribution in terms of the roots of $z^s = A(z)$ outside the unit circle, assuming that $A(z)$ is a polynomial. Chaudhry & Kim [7] used the same technique and presented some numerical work.

The technique of finding roots to complete a transform has become a classic one in queueing theory. It started from the analysis of the $M/D/s$ queue by Crommelin [10], whose solution required finding the roots within the unit circle of $z^s = e^{\lambda(z-1)}$ for some value $\lambda < s$. Through the years, root-finding turned out to be particularly important in the theory of bulk queues, originating from the work of Bailey [2] and Downton [13], who consider a bulk service queue with Poisson arrivals. For an overview on bulk queues we refer to Powell [24] and Chaudhry & Templeton [8].

Initially, the potential difficulties of root-finding were considered to be a slur on the unblemished transforms, since the determination of the roots can be numerically hazardous and the roots themselves have no probabilistic interpretation. However, Chaudhry and others [6] have made every effort to dispel the scepticism towards root-finding in queueing theory. They emphasize that root-finding in queueing is well-structured, in the sense that the roots are distinct for most models and that their location is well-predictable, so that numerical problems are not likely to occur.

While in general this is true for the moments of the stationary queue length, for determining the distribution itself, dependence on the roots might cause some problems, in particular for tail probabilities. In Chaudhry & Kim [7] a comparison is made between using the roots of $z^s = A(z)$ either inside or outside the unit circle. The performance of both approaches, though, heavily depends on the model parameters. Since it is therefore difficult to give a fair comparison, we choose to stress their common weakness: their performance inherently depends on how precise the roots of $z^s = A(z)$ are determined. Any deviation of the numerically determined roots from their true values results in errors in the computed probabilities.

The main purpose of this paper is to present an analytical rather than a numerical framework for dealing with the discrete-time bulk service queue. In particular, we will present explicit expressions for the roots of $z^s = A(z)$ and the stationary queue length distribution.

Under some mild conditions, we show that the roots of $z^s = A(z)$, both inside and outside the unit circle, can be represented as sample values of a periodic function with analytically given Fourier coefficients. In this way, the roots are no longer implicitly defined, and one can determine the roots as accurately as one wishes in a totally transparent way. Another way to determine the roots while maintaining transparency, results from applying successive substitution to a fixed-point equation. This idea originates from the work of Harris et al. [16] on root-finding for the continuous-time $G/E_k/1$ queue, and was presented more formally by Adan & Zhao [1] who distinguished a class of continuous distributions for which the method works. In this paper we further investigate the method for finding the roots of $z^s = A(z)$ for discrete distributions whose pgf is $A(z)$. We present necessary conditions for the method to work and compare these to the conditions needed for the Fourier series representation of the roots.

In deriving explicit formulas for the stationary queue length distribution, we first note that the discrete-time bulk service queue falls within the class of the $G/G/1$ queue. For the $G/G/1$ queue, the Laplace-Stieltjes transform of the stationary waiting time follows from Spitzer's identity (see e.g. [9]). We derive, using a Wiener-Hopf approach as in [9] to derive Spitzer's identity, a root-free expression for the pgf of the stationary queue length in the discrete-time bulk service queue. The pgf explicitly involves an infinite series of convolutions of A, and can be easily inverted, yielding explicit root-free expressions for the stationary queue length distribution. We remark that this is not the only way in which one can derive root-free expressions for the stationary distribution. An alternative approach would be to apply the matrix-analytical technique as introduced by Neuts [21]. In this approach, there is still the issue of the numerical determination of the so-called rate matrix, as the counterpart to the root-finding in the more traditional approach. Since we aim at providing explicit analytical expressions, we do not elaborate on the matrix-analytical technique in this paper.

In conclusion, our goals in this paper are:

(i) To present analytic formulas for the roots of $z^s = A(z)$.

(ii) To present a root-free and explicit representation of the stationary queue length distribution.

(iii) To demonstrate the numerical stability of the proposed method, in particular for tail probabilities.

2 The standard approach

In the above described discrete-time bulk service queue, the stationary queue length X, defined as

$$x_j = P(X = j) = \lim_{n\to\infty} P(X_n = j), \quad j = 0, 1, 2, \ \ldots\ , \tag{2}$$

exists under the assumption that $E(A) = \mu_A < s$. From the balance equations it then follows that the pgf of X is given by (see e.g. [3])

$$X(z) = \frac{A(z)\sum_{j=0}^{s-1} x_j(z^s - z^j)}{z^s - A(z)}, \tag{3}$$

which is assumed to be an analytic function in a disk $|z| \leq 1 + \epsilon$ with $\epsilon > 0$. The s unknowns $x_0, \ldots, x_{s-1}$ in the numerator of (3) can be determined by consideration of the zeros of the denominator of (3) that lie in the closed unit disk (see e.g. [2, 30]). With Rouché's theorem (see [9]), it can be shown that there are exactly s of these zeros. Thus by analyticity, the numerator of $X(z)$ should vanish at each of the zeros, yielding s equations. One of the zeros equals 1, and leads to a trivial equation. The normalization condition $X(1) = 1$ provides an additional equation.

We can, however, eliminate $x_0, \ldots, x_{s-1}$ from (3). Denoting the s roots of $z^s = A(z)$ in $|z| \leq 1$ by $z_0 = 1, z_1, \ldots, z_{s-1}$, (3) can be written as (see e.g. [3, 25])

$$X(z) = \frac{A(z)(z-1)(s-\mu_A)}{z^s - A(z)} \prod_{k=1}^{s-1} \frac{z - z_k}{1 - z_k}. \tag{4}$$

When $A(z)$ is a polynomial of degree $n > s$, (3) can be written as

$$X(z) = A(z) \prod_{k=s}^{n-1} \frac{1 - z_k}{z - z_k}, \tag{5}$$

where $z_s, z_{s+1}, \ldots, z_{n-1}$ are the $n - s$ roots of $z^s = A(z)$ outside the unit circle. Expressing the pgf of the stationary queue length for the discrete-time bulk service queue in terms of the roots outside the unit circle has been suggested by Zhao & Campbell [30], as stated above, although the idea stems from earlier work on bulk service queues by Bailey [2], Downton [13] and Chaudhry et al. [5].

2.1 Using roots inside the unit circle to compute x_j

We will now show how the stationary queue length distribution follows from (4). Let a_j denote the probability that A equals j, and recall that $a_0 > 0$.

From (4) we see that

$$X(z)(z^s - A(z)) =: cA(z)P(z), \tag{6}$$

where

$$c = \frac{s - \mu_A}{\prod_{k=1}^{s-1}(1 - z_k)}, \quad P(z) = \prod_{k=0}^{s-1}(z - z_k) = \sum_{j=0}^{s} p_j z^j. \tag{7}$$

Matching coefficients then gives for $j = 0, 1, \ldots$

$$x_j = \frac{1}{a_0} \sum_{n=1}^{j} (\delta_{n,s} - a_n) x_{j-n} - \frac{c}{a_0} \sum_{n=0}^{\min\{j,s\}} a_{j-n} p_n, \tag{8}$$

where $\delta_{n,s} = 1$ if $n = s$ and 0 otherwise.

2.2 Using roots outside the unit circle to compute x_j

Starting from (5), the following partial fraction expansion can be applied:

$$W(z) := \prod_{k=s}^{n-1} \frac{1 - z_k}{z - z_k} = \sum_{i=s}^{n-1} \frac{r_i}{z - z_i}, \tag{9}$$

where

$$\begin{aligned} r_i &= \lim_{z \to z_i} (z - z_i) W(z) \\ &= \frac{\prod_{k=s}^{n-1}(1 - z_k)}{\prod_{k=s, k \neq i}^{n-1}(z_i - z_k)}, \quad i = s, \ldots, n-1. \end{aligned} \tag{10}$$

When we rewrite (9) as

$$W(z) = -\sum_{k=0}^{\infty} \sum_{i=s}^{n-1} \left(\frac{r_i}{z_i}\right) \left(\frac{1}{z_i}\right)^k z^k, \tag{11}$$

it can be easily seen that the stationary queue length distribution is given by

$$x_j = -\sum_{k=0}^{j} a_k \sum_{i=s}^{n-1} \left(\frac{r_i}{z_i}\right) \left(\frac{1}{z_i}\right)^{j-k}, \quad j = 0, 1, 2, \ldots. \tag{12}$$

3 Analytic methods for finding the roots

We now pay further attention to the roots of $z^s = A(z)$. We first present an explicit expression for each of the roots as a Fourier series. Next, we elaborate on finding the roots using a fixed point iteration. We also point out how the conditions needed for the Fourier series representation and the fixed point iteration are related.

3.1 Fourier series representation

The roots of $z^s = A(z)$ lie on, what is called in [18], the generalized Szegö curve, defined by

$$\mathcal{S}_{A,s} := \{z \in \mathbb{C} \mid |z| \leq 1,\ |A(z)| = |z|^s\}. \tag{13}$$

For the notions used below from complex function theory we refer to [14, 26]. We impose the following condition:

Condition 3.1. *$\mathcal{S}_{A,s}$ is a Jordan curve with 0 in its interior, and $A(z)$ is zero-free on and inside $\mathcal{S}_{A,s}$.*

Recall that $a_0 > 0$ so that we have $|A(z)| > |z|^s$ for z in the interior of $\mathcal{S}_{A,s}$. Condition 3.1 is geometric in nature, and can be visually checked using some standard software package. A useful geometric formulation equivalent with Condition 3.1 is as follows:

Lemma 3.2. *Condition 3.1 is satisfied if and only if there is a Jordan curve J with $\mathcal{S}_{A,s}$ in its interior such that $A(z)$ is zero-free on and inside J while $|A(z)| < |z|^s$ on J.*

The proof that Condition 3.1 implies the existence of a J as in Lemma 3.2 uses continuity of A on $\mathcal{S}_{A,s}$ and some basic considerations of Jordan curve theory. The proof of the reverse implication can be based on the considerations in the proof of Lemma 3.3. For brevity we omit the details.

To present an equivalent form of Condition 3.1 of more analytic nature, we introduce the short-hand notation $C_{z^j}[f(z)]$ for the coefficient of z^j in $f(z)$. We have the following result:

Lemma 3.3. *Condition 3.1 is satisfied if and only if the coefficients $C_{z^{l-1}}[A^{l/s}(z)]$ decay exponentially in l.*

Proof. Assume that Condition 3.1 holds. Letting J as in Lemma 3.2 we see that we can define an analytic root $A^{1/s}(z)$ for z on and inside J that is positive at $z = 0$. We thus have by Cauchy's theorem

$$C_{z^{l-1}}[A^{l/s}(z)] = \frac{1}{2\pi i} \int_{z \in J} \frac{A^{l/s}(z)}{z^l}\, dz, \quad l = 1, 2, \ \ldots\ . \tag{14}$$

Since $|A(z)| < |z|^s$ for $z \in J$, it follows that

$$|C_{z^{l-1}}[A^{l/s}(z)]| \leq \frac{1}{2\pi} \text{length}(J) \Big(\max_{z \in J} \Big| \frac{A(z)}{z^s} \Big|^{1/s} \Big)^l, \tag{15}$$

and this decays exponentially, as required.

Now assume that $C_{z^{l-1}}[A^{l/s}(z)]$ decays exponentially. We shall sketch the proof that Condition 3.1 is valid; full details can be found in [18], proof of Lemma 4.1. We consider for w in a neighbourhood of 0 the equation

$$zA^{-1/s}(z) = w, \tag{16}$$

where we have taken in a neighbourhood of $z = 0$ the root $A^{-1/s}$ of A that is positive at $z = 0$ (recall $a_0 > 0$). By the Lagrange inversion theorem (see e.g [28], p. 133), the solution $z_0(w)$ of (16) has the power series representation

$$z_0(w) = \sum_{l=1}^{\infty} c_l w^l, \tag{17}$$

for w in a neighbourhood of 0 in which

$$c_l = \frac{1}{l!}\Big(\frac{d}{dz}\Big)^{l-1}\Big(\frac{z}{zA^{-1/s}(z)}\Big)^l\Big|_{z=0} = \frac{1}{l}C_{z^{l-1}}[A^{l/s}(z)]. \tag{18}$$

By assumption, we have that $c_l \to 0$ exponentially, whence the power series in (17) for $z_0(w)$ has a radius of convergence $R > 1$. It follows then from basic considerations in analytic function theory that $A^{-1/s}$ extends analytically to the open set $\{\sum_{l=1}^{\infty} c_l w_k^l \mid |w| < R\}$ and that $z_0(w)$ extends according to (17) on the set $|w| < R$. The Szegö set $\mathcal{S}_{A,s}$ in (13) occurs as

$$\mathcal{S}_{A,s} = \{z_0(e^{i\alpha}) \mid \alpha \in [0, 2\pi]\}, \tag{19}$$

and it can be shown that the parametrization

$$\alpha \in [0, 2\pi] \to z_0(e^{i\alpha}) = \sum_{l=1}^{\infty} c_l e^{il\alpha} \in \mathcal{S}_{A,s} \tag{20}$$

has no double points while a homotopy between $\{0\}$ and $\mathcal{S}_{A,s}$ is obtained according to

$$r \in [0, 1] \to \{z_0(re^{i\alpha}) \mid \alpha \in [0, 2\pi]\}. \tag{21}$$

From the latter facts it follows that $\mathcal{S}_{A,s}$ is a Jordan curve with 0 in its interior, and this completes the sketch of the proof of the converse statement. □

Note. The $z_0(w)$ of (17) is a univalent function, of a special type on an open set containing the closed unit disk $|w| \leq 1$. Hence, the results of the theory of univalent functions, as presented for instance in [14], Chs. 2-3, and [26], Ch. 12 become available. We shall not elaborate this point here, except for a casual note in Subsec. 3.2.

We now turn to the representation of the s roots of $z^s = A(z)$ in $|z| \leq 1$. These roots all lie inside the Jordan curve J in Lemma 3.2 and are given by

$$z_k = w_k A^{1/s}(z_k), \quad k = 0, 1, \ \ldots \ , s - 1, \tag{22}$$

where $w_k = e^{2\pi ki/s}$. Hence, from (20) we have

$$z_k = \sum_{l=1}^{\infty} c_l w_k^l, \quad k = 0, 1, \ \ldots \ , s - 1, \tag{23}$$

where c_l are explicitly given in (18).

When $A(z)$ is a polynomial of degree $n > s$, an expression similar to (23) can be derived for the $n - s$ roots of $z^s = A(z)$ outside the unit circle. Substituting $1/v$ into $z^s = A(z)$ and multiplying by v^n, we get

$$v^{n-s} = B(v), \tag{24}$$

where $B(v) = v^n A(1/v)$ is a polynomial of degree n. Note that from $|A(z)| < |z|^s$ for $1 < |z| < 1 + \delta$ for some $\delta > 0$, we have that $|B(v)| < |v|^{n-s}$ for $(1+\delta)^{-1} < |v| < 1$. Therefore, by Rouché's theorem, there occur exactly $n - s$ roots $v_k,\ k = \{s, s+1, \ldots, n-1\}$ of (24) in $|v| \le (1+\delta)^{-1}$, obviously satisfying $v_k = 1/z_k$. When there exists a Jordan curve (within $|v| < 1$) such that $B(v)$ is zero-free on and inside this curve, while 0 lies inside this curve and $|B(v)| < |v|^{n-s}$ on this curve, we find as above that

$$v_k = \sum_{l=1}^{\infty} \frac{1}{l} C_{v^{l-1}}[B^{l/(n-s)}(v)]\, e^{2\pi(k-s)il/(n-s)}, \quad k = s, s+1,\ \ldots\ , n-1. \tag{25}$$

The Condition 3.1 and its equivalent forms as given per Lemmas 3.2 and 3.3 are equally useful in deciding whether a given A satisfies it. We present now some instances where Condition 3.1 is satisfied.

i. $A(z)$ is zero-free in $|z| \le 1$. An appropriate Jordan curve J is found as $|z| = 1 + \delta$ with sufficiently small $\delta > 0$. Indeed, the assumptions on A imply that there is a $\delta > 0$ such that $0 < |A(z)| < |z|^s$ for $1 < |z| \le 1 + \delta$.

ii. $A(z)$ is zero-free in $|z| < 1$. There may occur now a finite number of zeros of A on $|z| = 1$, necessitating a modification of the Jordan curve J in (i). We indent this J around the zeros such that the zeros are outside the new J while $|A(z)| < |z|^s$ for all z on the new J. As one sees, this technique may also work in cases where there are zeros of A strictly inside $|z| = 1$. A class of examples follows from Kakeya's theorem [20] as follows:

   - when $a_0 > a_1 > \ldots$, we have that $A(z)$ is zero-free in $|z| \le 1$,
   - when $a_0 \ge a_1 \ge \ldots$, we have that $A(z)$ is zero-free in $|z| < 1$.

iii. The c_l in (18) are all non-negative. It follows from Pringsheim's theorem [27] and the fact that $z_0(w)$ is well-defined for $w \in [0, 1+\delta]$ with some $\delta > 0$, that the radius of convergence of the power series in (17) exceeds 1. Thus Lemma 3.3 applies and it follows that Condition 3.1 is satisfied.

Below we give two examples where one can compute the $c_l = C_{z^{l-1}}[A^{l/s}(z)]$ explicitly, so that the criterion in Lemma 3.3 can be verified.

Example 3.4. Consider the Poisson case, $a_j = e^{-\lambda}\lambda^j/j!,\ j = 0, 1, \ldots$, and $A(z) = \exp(\lambda(z-1))$ with $0 \le \theta := \lambda/s < 1$. In this case, Condition 3.1 is always satisfied. Furthermore, there holds that

$$c_l = e^{-l\theta}\,\frac{(l\theta)^{l-1}}{l!}. \tag{26}$$

In Fig. 1 we have pictured $\mathcal{S}_{A,s}$ for $\theta = 0.1, 0.5, 1.0$. The dots on the curves indicate the roots z_k for the case $s = 20$, obtained by calculating the sum in (23) up to $l = 50$.

Example 3.5. Consider the binomial case, $a_j = \binom{n}{j} q^j (1-q)^{n-j},\ j = 0, \ldots, n$, and $A(z) = (p + qz)^n$ where $p,\ q \ge 0,\ p + q = 1$ and $A'(1) = nq < s$. We compute in this case

$$c_l = \frac{1}{l}\, p^{l\beta - l + 1}\, q^{l-1} \binom{l\beta}{l-1}, \quad l = 1, 2,\ \ldots \tag{27}$$

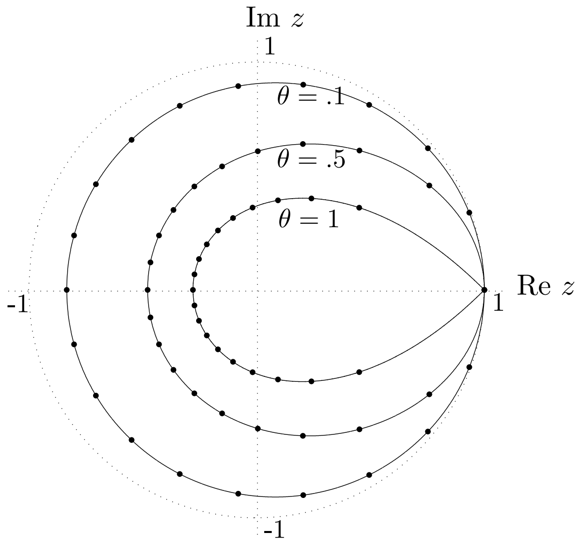

Figure 1: $\mathcal{S}_{A,s}$ for Poisson case, $\theta = .1, .5, 1$. The dots indicate $z_0, \ldots, z_{19}$ for $s = 20$, obtained by calculating the sum in (23) up to $l = 50$.

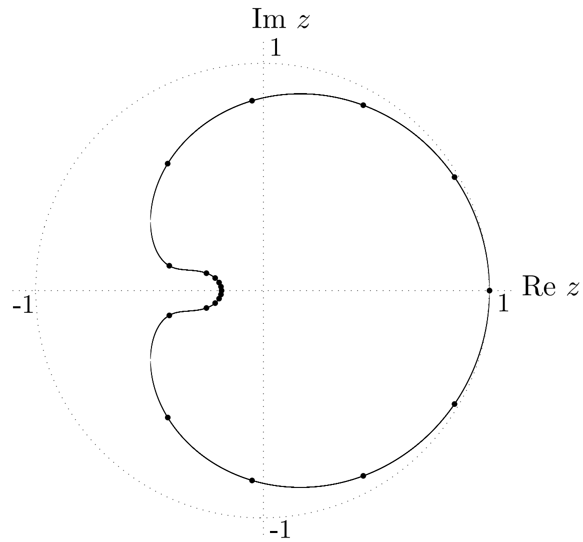

Figure 2: $\mathcal{S}_{A,s}$ for binomial case, $\beta = 0.5$, $q = .82$. The dots indicate $z_0, \ldots, z_{19}$ for $s = 20$, obtained by calculating the sum in (23) up to $l = 50$.

where $\beta := n/s$. In [18] the c_l are shown to have exponential decay for $\beta \geq 1$ (which covers in fact all practically relevant instances). It is further shown that for $0 \leq \beta < 1$ the c_l have exponential decay if and only if

$$p^{\beta-1} q (1-\beta)^{1-\beta} \beta^{\beta} < 1. \tag{28}$$

For $\beta = 1/2$, $s = 20$, constraint (28) requires q to be less than $2(\sqrt{2}-1)$. In Fig. 2 we plotted the $\mathcal{S}_{A,s}$ for $q = 0.82 < 2(\sqrt{2}-1)$, and the dots indicate the roots z_k obtained by calculating the sum in (23) up to $l = 50$. When q is increased, such that $q > 2(\sqrt{2}-1)$, $\mathcal{S}_{A,s}$ turns from a smooth Jordan curve containing zero into two separate closed curves (see [18]), and (23) no longer holds.

For the Poisson and binomial distribution we have (26) and (27), respectively, to determine the c_l. In general, the values of the c_l can be determined using the following property:

Property 3.6. *For $A(z) = \sum_{j=0}^{\infty} a_j z^j$ and $\alpha \in \mathbb{R}$, and $A^{\alpha}(z) = \sum_{j=0}^{\infty} b_j z^j$, the coefficients b_j follow from the coefficients a_j according to $b_0 = a_0^{\alpha}$ and*

$$b_{j+1} = \alpha a_0^{\alpha-1} a_{j+1} + \frac{1}{(j+1)a_0} \sum_{n=0}^{j-1} [\alpha(n+1) - (j-n)] a_{n+1} b_{j-n}, \quad j = 0, 1, \ldots. \tag{29}$$

The proof of Property 3.6 consists of computing the b_j's successively by equating coefficients in $A(z)(A^{\alpha})'(z) = \alpha A'(z) A^{\alpha}(z)$.

In [6] it is shown that the condition that A is infinitely-divisible, or the somewhat weaker condition that $A(z)$ has no zeros inside the unit circle, are sufficient for the roots of $z^s = A(z)$ on and within the unit circle to be distinct. However, examples exist of $A(z)$ having zeros inside the unit circle and at the same time having distinct roots (see e.g. Example 3.5). It is therefore that in both [6] and [16] the urge of finding a necessary condition for distinctness is expressed. In this respect, we have the following result:

Lemma 3.7. *When Condition 3.1 is satisfied, the roots of $z^s = A(z)$ on and within the unit circle are distinct.*

Proof. The roots lie inside J, and satisfy (22). Since $|A(z)|^{1/s} < |z|$ for all $z \in J$, it follows from Rouché's theorem that for each w_k, the function $z - w_k A^{1/s}(z)$ has as many zeros inside J as z. □

Although Condition 3.1 is not necessary for the roots to be distinct (as appears to be the case in Example 3.5 with $\beta = 1/2$ and $q = 0.83$), it covers a far larger class of distributions of A than those for which $A(z)$ has no zeros within the unit circle.

3.2 Fixed point iteration

We now discuss a way to determine the roots by applying successive substitution to a fixed-point equation. This idea originates from the work of Harris et al. [16] on root-finding for the continuous-time $G/E_k/1$ queue, and was presented more formally by Adan & Zhao [1] who distinguished a class of continuous random variables for which the method works. We further investigate the method for discrete random variables A. We present necessary conditions for the method to work and compare these to the conditions needed for the Fourier series representation of the roots introduced in the previous section.

When $A(z)$ is assumed to have no zeros for $|z| \leq 1$, we know that the s roots of $z^s = A(z)$ in $|z| \leq 1$ satisfy

$$z = wG(z), \tag{30}$$

with $G(z) = A^{1/s}(z)$ and $w^s = 1$. For each feasible w, Equation (30) can be shown as in Lemma 3.7 to have one unique root in $|z| \leq 1$. One could try to solve the equations by successive substitutions (see [1, 16]) as

$$z_k^{(n+1)} = w_k G(z_k^{(n)}), \quad k = 0, 1, ..., s-1, \tag{31}$$

with starting values $z_k^{(0)} = 0$.

Lemma 3.8. *When for $|z| \leq 1$, $A(z)$ is zero-free and $|G'(z)| < 1$, the fixed point equations (31) converge to the desired roots.*

Proof. For $|z| \leq 1$, $|w| \leq 1$,

$$|wG(z)| \leq G(|z|) \leq G(1) = 1, \tag{32}$$

so $wG(z)$ maps $|z| \leq 1$ into itself. For $|\tilde{z}|, |\hat{z}| \leq 1$ we have that

$$|wG(\tilde{z}) - wG(\hat{z})| \leq |\tilde{z} - \hat{z}| \max_{0 \leq t \leq 1} |G'(\hat{z} + t(\tilde{z} - \hat{z}))|. \tag{33}$$

Hence, from (33) and $|G'(z)| < 1$ for all $|z| \leq 1$, we conclude that $wG(z)$ is a contraction on $|z| \leq 1$. □

For the Poisson distribution with $\lambda < s$, it is readily seen that $A(z) \neq 0$ and $|G'(z)| < 1$ for $|z| \leq 1$, so that the iteration (31) works. We want to consider, however, also distributions for which $A(z)$ has zeros within the unit circle (see e.g. Example 3.5). We restrict here naturally to $A(z)$ that allow a root $G(z) = A^{1/s}(z)$ that is analytic around $\mathcal{S}_{A,s}$ and positive at 0. Hence we introduce the following condition:

Condition 3.9. *Condition 3.1 should be satisfied and for all points $z \in \mathcal{S}_{A,s}$ there should hold that $|G'(z)| < 1$.*

According to the maximum principle we have that Condition 3.9 implies that $|G'(z)| < 1$ holds for all points inside $\mathcal{S}_{A,s}$ as well. Condition 3.9 thus ensures that for $\alpha \in [0, 2\pi]$ the point z_k is an attractor for the iteration (31).

Note that Condition 3.9 is what is minimally needed to ensure (31) to converge locally. However, under Condition 3.9 the iterates are by no means guaranteed to stay in $\mathcal{S}_{A,s}$ and its interior. This is already seen for the binomial case with $\beta < 1$, s even, and the iteration (31) for $k = s/2$, i.e

$$z_{s/2}^{(n+1)} = -1(p + qz_{s/2}^{(n)})^{\beta}. \tag{34}$$

For this iteration, the $z_{s/2}^{(n)}$, $n = 0, 1, \ldots$, are alternatingly inside and outside $\mathcal{S}_{A,s}$. The iteration, though, converges to the correct point when q is not too large. It is difficult, in general, to give guarantees for convergence; nevertheless, convergence seems to occur in most cases where Condition 3.9 holds.

While Condition 3.1 implies that $\mathcal{S}_{A,s}$ is a closed curve without double points, Condition 3.9 apparently does not hold for all such curves. We shall compare Condition 3.9 with the notions convexity and starshapedness from the geometric theory of univalent functions.

Definition 3.10. *(i) A closed curve without double points is called starshaped with respect to a point in its interior if any ray from this point intersects the curve at exactly one point,*

(ii) A closed curve without double points is called convex when it is starshaped with respect to any point in its interior.

We have the following result.

Theorem 3.11. *There holds*

$$\mathcal{S}_{A,s} \text{ is convex } \Rightarrow \text{ Condition 3.9 holds } \Rightarrow \mathcal{S}_{A,s} \text{ is starshaped with respect to 0.} \tag{35}$$

Proof. See Appendix A. □

Example 3.5 gives a nice demonstration of Thm. 3.11. From an inspection of $\mathcal{S}_{A,s}$ in Fig. 2 one sees that $\mathcal{S}_{A,s}$ is not starshaped with respect to 0, and one can thus immediately conclude that Condition 3.9 is not satisfied and hence the iteration (31) cannot be applied to determine the roots.

Example 3.12. Consider the binomial case, $A(z) = (p + qz)^n$ where p, $q \geq 0$, $p + q = 1$ and $\beta = n/s$. When $\beta \geq 1$, we have that

$$\max_{z \in \mathcal{S}_{A,s}} |G'(z)| = \max_{z \in \mathcal{S}_{A,s}} |\beta q(p + qz)^{\beta - 1}|, \tag{36}$$

occurs at $z = 1$ and equals $\beta q < 1$. For $\beta \in (0, 1)$, it can be shown that Condition 3.9 holds if and only if

$$p^{\beta - 1} q (1 + \beta)^{1 - \beta} \beta^{\beta} < 1. \tag{37}$$

Now denote by $q_1(\beta)$ and $q_2(\beta)$ the values of q for which $<$ can be replaced by $=$ in (28) and (37), respectively. These values, that can be shown to be unique, are plotted in Fig. 3. Observe that the set of values q for which the Fourier series representation holds ($q < q_1(\beta)$)

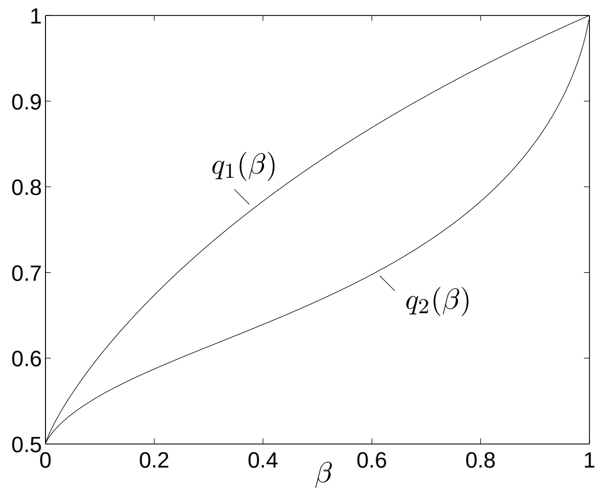

Figure 3: $q_1(\beta)$ and $q_2(\beta)$ for $\beta \in (0, 1)$.

is much larger than the set for which Condition 3.9 holds ($q < q_2(\beta)$). We have numerical evidence that whenever $q < q_2(\beta)$ the iteration (31) works. Finally note that the roots in Fig. 2 ($\beta = 1/2, q = 0.82$), which are computed using the Fourier series representation, cannot be obtained using the fixed point iteration. To compare the condition of convexity and Condition 3.9 we just consider the case $\beta = 1/2$: when $2 - \sqrt{2} < q < 2/3$ we have that $\mathcal{S}_{A,s}$ satisfies Condition 3.9 while $\mathcal{S}_{A,s}$ is not convex at the intersection point with the negative real axis.

Although the fixed point iteration is a very efficient method, the class of distributions of A for which it can be applied is clearly smaller than the class of distributions of A for which the Fourier series representation holds. That is, Condition 3.1 is much weaker than Condition 3.9.

3.3 Numerical results

We will now present two examples for which the roots can be determined through (31). For each root we stop the iteration when

$$|z_k^{(n+1)} - z_k^{(n)}| < 10^{-14}, \tag{38}$$

and we denote the resulting values by $\hat{z}_k$.

Denote by $z_k(L)$ the estimated root value that results when we truncate l at L in (23). We would like to have some more insight in how fast $z_k(L)$ converges to z_k, where the $\hat{z}_k$ determined above are considered to be sufficiently accurate approximations of the z_k to serve as references.

For the Poisson distribution with $\lambda = 8$, $s = 10$, Table 1 displays the roots $\hat{z}_k$, along with the distance between $\hat{z}_k$ and $z_k(L)$ for $L = 10, 20, 50$. As it appears, the Fourier series (23) converge quite rapidly. We further note that the most slowly convergent series is $z_0(L)$. This can be explained from the following result:

Lemma 3.13. *The truncation error $|z_k(L) - z_k|$ is largest for $k = 0$ among all $k = 0, 1, \ldots, s-1$, when the coefficients $c_l \geq 0$, $l = 1, 2, \ldots$.*

Proof. Follows directly from (23). □

For the Poisson case the c_l are greater or equal to 0, indeed. This is also the case for e.g. the geometric distribution $a_j = (1-p)p^j$ with $0 \le p < 1$, and for the binomial distribution in Example 3.5 with $\beta \ge 1$, but it fails to hold for the latter distribution with $0 < \beta < 1$.

In general, if one applies (23) to a distribution A for which $c_l \ge 0$, then $|z_0(L) - z_0| = |z_0(L) - 1|$ being small is a good test for convergence, since it reflects the maximum distance between the estimated and true values of the roots.

Table 2 displays $\hat{z}_k$ and $|z_k(L) - \hat{z}_k|$, $L = 10, 20, 50$, for the binomial distribution with $n = 16$, $q = 0.5$ and $s = 10$, so that all $c_l \ge 0$.

Table 1: Poisson distribution, $\lambda = 8$, $s = 10$. The roots of $z^s = A(z)$ for $|z| \le 1$ determined with (31) (denoted as $\hat{z}_k$), along with the distance between $\hat{z}_k$ and $z_k(L)$ for $L = 10, 20, 50$.

k	Re $\hat{z}_k$	Im $\hat{z}_k$	$\lvert z_k(10) - \hat{z}_k\rvert$	$\lvert z_k(20) - \hat{z}_k\rvert$	$\lvert z_k(50) - \hat{z}_k\rvert$
0	1.000000	0.000000	0.110194	0.048179	0.009637
1	0.300438	0.486051	0.017461	0.005283	0.000694
2	−0.017701	0.442657	0.009539	0.002817	0.000366
3	−0.205881	0.320697	0.006988	0.002052	0.000266
4	−0.308844	0.166704	0.005961	0.001747	0.000226
5	−0.341824	0.000000	0.005673	0.001662	0.000215

Table 2: Binomial distribution, $n = 16$, $q = 0.5$, $s = 10$. The roots of $z^s = A(z)$ for $|z| \le 1$ determined with (31) (denoted as $\hat{z}_k$), along with the distance between $\hat{z}_k$ and $z_k(L)$ for $L = 10, 20, 50$.

k	Re $\hat{z}_k$	Im $\hat{z}_k$	$\lvert z_k(10) - \hat{z}_k\rvert$	$\lvert z_k(20) - \hat{z}_k\rvert$	$\lvert z_k(50) - \hat{z}_k\rvert$
0	1.000000	0.000000	0.118685	0.037943	0.003067
1	0.169044	0.439341	0.024368	0.006010	0.000364
2	−0.066258	0.315413	0.013329	0.003216	0.000192
3	−0.164522	0.199596	0.009766	0.002344	0.000140
4	−0.208378	0.096590	0.008330	0.001996	0.000119
5	−0.221147	0.000000	0.007928	0.001899	0.000113

We stress that there are many distributions of A for which the iteration (31) fails to work, while (23) still holds, i.e. Condition 3.1 is satisfied (see e.g Example 3.5 and 4.5). We simply chose the above examples so that we could obtain precise estimates of the real roots without invoking some other, less transparent, numerical method than (31).

4 The general approach

To find explicit expressions for the stationary queue length distribution, we start from the observation that the queue length at the beginning of slot n in (1) can be viewed as being the sojourn time of the n-th customer in a queue for which s equals the deterministic and integer-valued interarrival time between customer n and $n+1$, and A_n is the service time of customer $n+1$. This model is also referred to as the $D/G/1$ queue (see e.g. Servi [25]), and falls within the class of the $G/G/1$ queue. For the $G/G/1$ queue, the Laplace transform of the stationary waiting time follows from Spitzer's identity (see e.g. [9]). Using similar arguments as used for the derivation of Spitzer's identity, we can prove the following result:

Theorem 4.1. *The pgf of the stationary queue length distribution is given by*

$$X(z) = A(z)\exp\Big\{-\sum_{l=1}^{\infty}\frac{1}{l}P(S_l>0)\Big\}\exp\Big\{\sum_{l=1}^{\infty}\frac{1}{l}E(z^{S_l}\mathbf{1}\{S_l>0\})\Big\}, \tag{39}$$

where $S_l = \sum_{i=1}^{l}(A_i - s)$*, and* $\mathbf{1}\{B\} = 1$ *if* B *holds and 0 otherwise.*

Proof. This proof is based on Wiener-Hopf decomposition, analogously to [9] p. 338 for the continuous-time case. From recursion (1) we have

$$\begin{aligned} E(z^{X_{t+1}}) &= E(z^{A_t}\mathbf{1}\{X_t \le s\}) + E(z^{X_t+A_t-s}\mathbf{1}\{X_t>s\}) \\ &= P(X_t\le s)E(z^{A_t}) + E(z^{X_t+A_t-s}) - E(z^{X_t+A_t-s}\mathbf{1}\{X_t\le s\}). \end{aligned} \tag{40}$$

Letting $t\to\infty$ and observing that X_t and A_t are independent then yields

$$\frac{X(z)}{A(z)}(1-z^{-s}A(z)) = P(X\le s) - E(z^{X-s}\mathbf{1}\{X\le s\}). \tag{41}$$

We denote the right-hand side of (41) as $X^*(z)$. Using

$$\frac{1}{1-z} = \exp\{-\ln(1-z)\} = \exp\Big\{\sum_{l=1}^{\infty}\frac{z^l}{l}\Big\}, \quad |z|<1, \tag{42}$$

we have that

$$\begin{aligned} (1-z^{-s}A(z))^{-1} &= \exp\Big\{\sum_{l=1}^{\infty}\frac{1}{l}(z^{-s}A(z))^l\Big\} \\ &= \exp\Big\{\sum_{l=1}^{\infty}\frac{1}{l}E(z^{S_l}\mathbf{1}\{S_l>0\})\Big\}\cdot\exp\Big\{\sum_{l=1}^{\infty}\frac{1}{l}E(z^{S_l}\mathbf{1}\{S_l\le 0\})\Big\}. \end{aligned} \tag{43}$$

Substituting (43) into (41) yields

$$\frac{X(z)}{A(z)}\exp\Big\{-\sum_{l=1}^{\infty}\frac{1}{l}E(z^{S_l}\mathbf{1}\{S_l>0\})\Big\} = X^*(z)\exp\Big\{\sum_{l=1}^{\infty}\frac{1}{l}E(z^{S_l}\mathbf{1}\{S_l\le 0\})\Big\}. \tag{44}$$

The left-hand side and right-hand side of (44) are analytic and bounded in $|z|<1$ and $|z|>1$, respectively, and continuous up to $|z|=1$ (see [9] p. 338). Therefore, their analytic continuation contains no singularities in the entire complex plane, whence upon using Liouville's theorem (see e.g. [28]) the left-hand side of (44) is constant, i.e.

$$\frac{X(z)}{A(z)} = K\exp\Big\{\sum_{l=1}^{\infty}\frac{1}{l}E(z^{S_l}\mathbf{1}\{S_l>0\})\Big\}. \tag{45}$$

The constant K follows from $X(1)/A(1)=1$ yielding

$$K = \exp\Big\{-\sum_{l=1}^{\infty}\frac{1}{l}P(S_l>0)\Big\}, \tag{46}$$

which completes the proof. □

In principle, the mean and variance of the stationary queue length can be determined from the x_j. However, this can be done more directly from (39), due to $\mu_X = X'(1)$ and $\sigma_X^2 = X''(1)+X'(1)-X'(1)^2$. Let B_l denote the random variable that is distributed according to the l-fold convolution of the distribution of A. We then have that

$$\mu_X = \mu_A + \sum_{l=1}^{\infty} \frac{1}{l} \sum_{i=ls}^{\infty} (i-ls)P(B_l = i), \tag{47}$$

$$\sigma_X^2 = \sigma_A^2 + \sum_{l=1}^{\infty} \frac{1}{l} \sum_{i=ls}^{\infty} (i-ls)^2 P(B_l = i), \tag{48}$$

which are root-free expressions for μ_X and σ_X^2. For comparison, we mention that taking derivatives of (4) instead of (39) yields, upon a lengthy calculation,

$$\mu_X = \frac{\sigma_A^2}{2(s-\mu_A)} + \frac{1}{2}\mu_A - \frac{1}{2}(s-1) + \sum_{k=1}^{s-1} \frac{1}{1-z_k}, \tag{49}$$

$$\begin{aligned}\sigma_X^2 = \sigma_A^2 \; &+ \; \frac{A'''(1)-s(s-1)(s-2)}{3(s-\mu_A)} + \frac{A''(1)-s(s-1)}{2(s-\mu_A)} \\ &+ \; \Big(\frac{A''(1)-s(s-1)}{2(s-\mu_A)}\Big)^2 - \sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2}.\end{aligned} \tag{50}$$

4.1 Stationary queue length distribution

From (39) the following is readily seen:

Lemma 4.2. *The stationary queue length distribution is given by*

$$x_j = d \sum_{k=0}^{j} a_k \, C_{z^{j-k}} \Big[\exp\Big\{\sum_{l=1}^{\infty}\sum_{i=1}^{\infty} \frac{1}{l} \, P(B_l = ls+i) z^i\Big\}\Big], \quad j = 0,1,\ldots, \tag{51}$$

and

$$d = \exp\Big\{-\sum_{l=1}^{\infty} \sum_{i=ls+1}^{\infty} \frac{1}{l} \, P(B_l = i)\Big\}. \tag{52}$$

We showed in [18] that (51) can be alternatively derived using Fourier sampling. We note here that He and Sohraby [17] also derived a root-free expression of a similar type for the stationary queue length distribution, using combinatorial arguments and Ballot theorems.

Expression (51) provides for each x_j a representation that does not depend on the roots of $z^s = A(z)$. However, the series contain two infinite series. So, in working with (51) we should have some feeling for the speed of convergence of these series in relation with choosing appropriate truncation levels for the sums. This issue is interesting in its own right, and has been further investigated by the authors in [19]. Some guidelines for truncating the series are given below.

For determining the coefficients in (51) we can use the following property:

Property 4.3. *For* $A(z)=\sum_{j=0}^{\infty}a_jz^j$ *and* $B(z)=\sum_{j=0}^{\infty}b_jz^j$ *for which it holds that* $A(z)=\exp\{B(z)\}$*, the coefficients* a_j *follow recursively from the coefficients* b_j *(and vice versa) according to*

$$a_0=\exp(b_0);\quad a_j=\frac{1}{j}\sum_{n=1}^{j}nb_na_{j-n},\quad j=1,2,\ldots. \tag{53}$$

The proof of Property 4.3 consists of computing the a_j's successively by equating coefficients in $A'(z)=B'(z)A(z)$.

4.2 Some guidelines for numerical work

First note that when $A(z)$ is a polynomial of degree n, the sum in (51) over i would be finite, running from $i=1$ to $i=ln$. For non-polynomial $A(z)$, we should truncate A at n such that $P(A>n)$ is negligible.

We further mention that determining d is the bottleneck, which can be seen in the following way. Denote by $x_j(L)$ and $d(L)$ the estimated value of x_j and d that result from truncating the sum over l at $l=L$ in (51) and (52), respectively. The relative error made then equals

$$\begin{aligned}\frac{d(L)-d}{d} &= \exp\Big\{\sum_{l=L+1}^{\infty}\sum_{i=ls+1}^{\infty}\frac{1}{l}\,P(B_l=i)\Big\}-1\\ &\approx \sum_{l=L+1}^{\infty}\sum_{i=ls+1}^{\infty}\frac{1}{l}\,P(B_l=i) = \sum_{l=L+1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{l}\,P(B_l=ls+i),\end{aligned} \tag{54}$$

where the far right-hand side of (54) sums all truncation errors $\sum_{l=L+1}^{\infty}\frac{1}{l}\,P(B_l=ls+i)$ that appear in (51) when computing x_j.

A good check for overall accuracy is provided by the relation (see [18])

$$d=\sum_{j=0}^{s}x_j, \tag{55}$$

from which it follows that

$$\sum_{i=0}^{s}\sum_{k=0}^{j}a_k\,C_{z^{j-k}}\Big[\exp\Big\{\sum_{l=1}^{L}\sum_{i=1}^{\infty}\frac{1}{l}\,P(B_l=ls+i)z^i\Big\}\Big]\uparrow 1,\quad L\to\infty. \tag{56}$$

Hence, when the above quantity is close enough to 1, the accuracy of the estimated values seems guaranteed.

4.3 Numerical results

Example 4.4. Consider the Poisson case, $A(z)=\exp(\lambda(z-1))$, $\lambda<s$. For $s=10$ and $\lambda=5,8,9$, Table 3 displays the mean and variance of the stationary queue length. For the results obtained from (49) and (50) we have determined the roots using (31) with stopping criterion $|z_k^{(n+1)}-z_k^{(n)}|<10^{-14}$. For the results obtained from (47) and (48) we have truncated the sum over l at $l=30$ and the sum over i at $i=300$. We observe that the higher the load, the higher we should choose the level at which we truncate these sums. For $\lambda=5$ and $\lambda=8$ the truncation levels chosen are sufficient, while for $\lambda=9$ they should be taken somewhat higher.

Table 3: Mean and variance of X for the Poisson case with $s = 10$, $\lambda = 5, 8, 9$.

	μ_X		σ_X^2	
	(49)	(47)	(50)	(48)
λ=5	5.0237	5.0237	5.0519	5.0519
λ=8	8.8786	8.8786	11.6109	11.6104
λ=9	12.1012	11.9938	29.9067	28.4381

Example 4.5. We take the example considered in [7], in which $A(z) = Y(z)^6$ where

$$Y(z) = 0.1 + 0.15z + 0.2z^2 + 0.2z^3 + 0.15z^4 + 0.1z^5 + 0.05z^6 + 0.01z^7 + 0.01z^8 + 0.03z^{10}. \quad (57)$$

In [7] the stationary queue length distribution is determined from (12), for which the zeros outside the unit circle are determined numerically using the computer package QROOT. The iteration (31) does not work for this example. We calculate the stationary queue length distribution from (8), (12) and (51). For (8) and (12) we calculate the roots of $z^s = A(z)$ inside and outside the unit circle using (23) and (25), respectively. For (23), (25) and (51) we truncate the sum over l at $l = 60$. The results are displayed in Table 4. We see that both

Table 4: Stationary queue length distribution for $A(z) = Y(z)^6$, with $Y(z)$ given in (57), $s = 30$.

j	x_j from [7]	x_j from (8)-(23)	x_j from (12)-(25)	x_j from (51)
0	0.00000098	0.00000098	0.00000098	0.00000098
1	0.00000885	0.00000885	0.00000885	0.00000885
2	0.00004501	0.00004500	0.00004501	0.00004501
3	0.00016686	0.00016680	0.00016686	0.00016686
4	0.00049733	0.00049715	0.00049733	0.00049733
5	0.00125555	0.00125503	0.00125555	0.00125555
6	0.00277138	0.00277010	0.00277138	0.00277138
7	0.00546268	0.00545988	0.00546269	0.00546268
8	0.00976060	0.00975504	0.00976060	0.00976060
9	0.01598541	0.01597540	0.01598541	0.01598541
10	0.02420260	0.02418598	0.02420260	0.02420260
20	0.06498585	0.06487376	0.06498585	0.06498585
30	0.00728773	0.00661255	0.00728773	0.00728773
40	0.00015022	0.00049559	0.00015022	0.00015022
50	0.00000080	0.00072575	0.00000080	0.00000080

(12) and (51) lead to similar results as obtained in [7]. Determining the probabilities from (8) gives problems when moving into the tail of the distribution. Although these problems might be resolved by truncating the sum over l in (23) at a higher level, (12) and (51) seem more stable. The truncation level of $l = 60$ is sufficient, although it is no problem to increase it from a numerical point of view.

Example 4.6. Consider the binomial case, $A(z) = (p + qz)^n$ where p, $q \geq 0$, $p + q = 1$, for which we take $n = 16$, $q = 0.5$, $s = 10$. Table 5 displays some of the x_j, calculated by $x_j(L)$ for $L = 10, 20, 30$. Additionally, the x_j have been determined from (12) where the roots of $z^s = A(z)$ outside the unit circle follow from (25) (with the sum over l truncated at $l = 60$). Note that for x_{50} and x_{100} we need some higher level of L to determine these small

Table 5: Stationary queue length distribution for the binomial case, $n = 16$, $q = 0.5$, $s = 10$.

j	$x_j(10)$	$x_j(20)$	$x_j(30)$	(12)
0	$0.13132067 \cdot 10^{-4}$	$0.13131227 \cdot 10^{-4}$	$0.13131225 \cdot 10^{-4}$	$0.13131228 \cdot 10^{-4}$
10	$0.12967227 \cdot 10^{-0}$	$0.12967413 \cdot 10^{-0}$	$0.12967413 \cdot 10^{-0}$	$0.12967413 \cdot 10^{-0}$
20	$0.10032061 \cdot 10^{-4}$	$0.10120244 \cdot 10^{-4}$	$0.10120578 \cdot 10^{-4}$	$0.10120543 \cdot 10^{-4}$
30	$0.25745593 \cdot 10^{-9}$	$0.29484912 \cdot 10^{-9}$	$0.29527164 \cdot 10^{-9}$	$0.29527217 \cdot 10^{-9}$
50	$0.07585901 \cdot 10^{-18}$	$0.22301750 \cdot 10^{-18}$	$0.25004662 \cdot 10^{-18}$	$0.25112237 \cdot 10^{-18}$
70	$0.01219941 \cdot 10^{-27}$	$0.11512297 \cdot 10^{-27}$	$0.19133604 \cdot 10^{-27}$	$0.21357399 \cdot 10^{-27}$
100	$0.00126202 \cdot 10^{-41}$	$0.09314437 \cdot 10^{-41}$	$0.28000048 \cdot 10^{-41}$	$0.52971556 \cdot 10^{-41}$

probabilities up to a reasonable accuracy. Increasing L would give no numerical difficulties, so that the accuracy is just a matter of choice. This makes this approach well-suited for calculating tail probabilities.

5 Conclusions

In the commonly used approaches to the discrete-time bulk service queue, the stationary queue length follows from the roots inside or outside the unit disk of a characteristic equation. We have presented representations of these roots as Fourier series, making the classic approach transparent and explicit. The Fourier series are easy to implement and numerically stable.

We further presented analytic formulas for the stationary queue length distribution that do not depend on roots. This solution is explicit and well-suited for determining tail probabilities up to a high accuracy.

Acknowledgement

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A Proof of Theorem 3.11

We have $z(\alpha) = z_0(e^{i\alpha})$ where $z_0(w)$ is the solution of

$$z_0(w) = wG(z_0(w)), \quad G(z) = A^{1/s}(z), \tag{58}$$

see the proof of Lemma 3.3.

The notions of convexity and starshapedness vary slightly from one place to another in the literature. We use here the notions as can be found in [23], p. 125, but we shall also use results from [14] and [26] that use a somewhat different convention.

Assume that $\mathcal{S}_{A,s}$ is convex in the sense of [23]. According to [23], Exercise 108 on p.125, we have

$$\mathrm{Re}\Big[1 + \frac{wz_0''(w)}{z_0'(w)}\Big] > 0, \quad |w| = 1. \tag{59}$$

By continuity, the inequality (59) holds in an annulus $1 \le |w| < 1 + \delta$ for some $\delta > 0$. Then, by the theory as given in [26], Ch. 12, Sec. 2 (in particular, the material on pp. 335-336), we have that $z_0(w)$ is convex in $|w| < 1 + \delta$ in the sense of [26], Ch. 12 and [14], Ch. 2. Now by [14], Corollary 1(a) on p. 251, we have that

$$z_0(w) \text{ is convex in } |w| < 1 + \delta \ \Rightarrow\ \mathrm{Re}\Big[\frac{w z_0'(w)}{z_0(w)}\Big] > 1/2, \quad |w| < 1 + \delta. \tag{60}$$

From (58) we have upon computation

$$\frac{w z_0'(w)}{z_0(w)} = \frac{1}{1 - wG'(z_0(w))}. \tag{61}$$

Now note that for $v \in \mathbb{C}$, $v \neq 1$ we have

$$\mathrm{Re}\Big[\frac{1}{1-v}\Big] > \frac{1}{2} \ \Leftrightarrow\ |v| < 1. \tag{62}$$

Hence, when $z_0(w)$ is convex, there holds $|wG'(z_0(w))| < 1$ for $|w| < 1+\delta$ and thus Condition 3.9 holds.

Next assume that Condition 3.9 holds. To prove that $\mathcal{S}_{A,s}$ is starshaped with respect to the origin, it is sufficient by [23], Exercise 109 on p. 125, to show that

$$\phi'(\alpha) := \frac{\mathrm{d}}{\mathrm{d}\alpha}[\arg z(\alpha)] = \mathrm{Re}\Big[\frac{w z_0'(w)}{z_0(w)}\Big] > 0, \tag{63}$$

for $w = e^{i\alpha}$, $\alpha \in [0, 2\pi]$. Since $|wG'(z_0(w))| < 1$ by Condition 3.9, it is seen at once from (61) and (62) that $\phi'(\alpha) > 1/2 > 0$. This completes the proof. $\square$

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