SOME ITERATIVE ALGORITHMS TO COMPUTE CANONICAL WINDOWS FOR GABOR FRAMES

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We analyze some iterative algorithms for the computation of the canonical tight window g^t and the canonical dual window g^d associated with a Gabor frame (g, a, b). As to the computation of g^t, we consider algorithms that do require inversion of intermediate frame operators as well as algorithms that do not require inversions. As to the computation of g^d, we naturally consider algorithms where no frame operator inversions are required. These algorithms have safe but conservative versions, with guaranteed convergence of prescribed order but with suboptimal convergence constants, and smart but risky versions, with near-optimal convergence of prescribed order which is, however, guaranteed only if the frame bound ratio A/B of (g, a, b) exceeds an analytically given lower bound. Thus we propose for g^t an algorithm, using inversions, with quadratic convergence, and two algorithms, using no inversions, with quadratic and cubic convergence, respectively, and we identify for these algorithms the safe and the smart versions. For g^d we propose two algorithms, without inversions, with quadratic and cubic convergence, respectively, and also for these algorithms we identify the safe and the smart versions. All these algorithms can be formulated by using a general mechanism for proposing the recursion step in an approximation scheme for g^t and g^d with a prescribed error decay. The tools used to analyze the algorithms are the calculus of frame operators, the spectral mapping theorem, and Kantorovich's inequality.

Keywords: Gabor frame, canonical tight window, canonical dual window, iterative method, quadratic convergence, cubic convergence, spectral mapping theorem, Kantorovich's inequality.

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1. Introduction

We consider for $a > 0$, $b > 0$ and $g \in L^2(\mathbb{R})$ the Gabor system

$$(g, a, b) = (g_{na,mb})_{n,m\in\mathbb{Z}} \ , \tag{1}$$

where for $x, y \in \mathbb{R}$ we denote by

$$g_{x,y}(t) = e^{2\pi i y t}\, g(t - x) \ , \qquad t \in \mathbb{R} \ , \tag{2}$$

the time-frequency shifted version $g_{x,y}$ of g. We assume that (g, a, b) is a frame (so that it is assumed implicitly that $ab \leq 1$). We denote by S the frame operator

$$S : f \in L^2(\mathbb{R}) \to \sum_{n,m=-\infty}^{\infty} (f, g_{na,mb})\, g_{na,mb} \in L^2(\mathbb{R}) \ , \tag{3}$$

and we denote by A, B the best lower, upper frame bounds

$$A = \min \sigma(S) > 0 \ , \qquad B = \max \sigma(S) < \infty \ , \tag{4}$$

where $\sigma(S)$ is the spectrum of the positive, bounded operator S. We, furthermore, denote by g^t and g^d the canonical tight window and the canonical dual window associated with the frame (g, a, b) according to

$$g^t = S^{-1/2} g \ , \qquad g^d = S^{-1} g \ , \tag{5}$$

respectively.

We refer for extensive information on Gabor frames and the role of the canonical windows to [6], [1], Ch. 4, [2], [4], Chs. 5–9, and to Feichtinger's contribution [3] to the present IMS series Volume. In particular, we have that S, $S^{-1/2}$, S^{-1}, etc. commute with the relevant shift operators $f \in L^2(\mathbb{R}) \to f_{na,mb} \in L^2(\mathbb{R})$ with integer n, m. Furthermore, the triples (g^t, a, b) and (g^d, a, b) are themselves Gabor frames, with respective frame operators I and S^{-1}.

2. Overview

In this section we present a number of iterative algorithms to approximate g^t and g^d, and we describe the results obtained for these algorithms in this contribution. We let $\gamma_0 = g$ and for $k = 0, 1, \dots,$ we set

$$\text{I.} \quad \gamma_{k+1} = \tfrac{1}{2}\,\alpha_k\gamma_k + \tfrac{1}{2}\,\beta_k S_k^{-1}\gamma_k \ , \tag{6}$$

$$\text{II.} \quad \gamma_{k+1} = \tfrac{3}{2}\,\varepsilon_{k0}\gamma_k - \tfrac{1}{2}\,\varepsilon_{k1} S_k\gamma_k \ , \tag{7}$$

$$\text{III.} \quad \gamma_{k+1} = \tfrac{15}{8}\,\varepsilon_{k0}\gamma_k - \tfrac{5}{4}\,\varepsilon_{k1} S_k\gamma_k + \tfrac{3}{8}\,\varepsilon_{k2} S_k^2\gamma_k \ , \tag{8}$$

$$\text{IV.} \quad \gamma_{k+1} = 2\delta_{k0}\gamma_k - \delta_{k1} S_k g \ , \tag{9}$$

$$\text{V.} \quad \gamma_{k+1} = 3\delta_{k0}\gamma_k - 3\delta_{k1} S_k g + \delta_{k2} S S_k\gamma_k \ , \tag{10}$$

respectively. Here S_k is the frame operator corresponding to (γ_k, a, b), and the scalars $\alpha_k, \beta_k > 0$ in (6), ε_{ki} in (7)–(8) and δ_{ki} in (9)–(10) have to be chosen appropriately. In particular, it will be necessary to choose these scalars in such a way that (γ_{k+1}, a, b) is again a Gabor frame.

The algorithm with recursion step I and with

$$\alpha_k = \|\gamma_k\|^{-1} \ , \qquad \beta_k = \|S_k^{-1}\gamma_k\| \tag{11}$$

has been analyzed in [6]. It is shown there that there is quadratic and monotone convergence (in a sense that will be clear to the reader after Sec. 3.) of $\gamma_k/\|\gamma_k\|$ to $g^t/\|g^t\|$. Before we proceed it should be said that [6] contains a number of rather innocent but disturbing errors; a corrected version of [6] can be found at

`http://www.math.ucdavis.edu/~strohmer/papers/2000/tight.html`

In Sec. 4. we shall repeat the arguments given in [6].

The algorithms with recursion step II and IV will be analyzed in detail in the present contribution; they yield sequences γ_k with $\gamma_k/\|\gamma_k\| \to g^t/\|g^t\|$ and $\gamma_k/\|\gamma_k\| \to g^d/\|g^d\|$, respectively, when ε_{k0}, ε_{k1} and δ_{k0}, δ_{k1} are chosen appropriately, and the convergence is (at least) quadratic. These algorithms have safe but conservative versions and smart but risky versions, depending on the way ε_{k0}, ε_{k1} and δ_{k0}, δ_{k1} are chosen.

The safe but conservative versions are obtained by replacing

$$g \text{ by } g/\hat{B}^{1/2} \ , \qquad S \text{ by } S/\hat{B} \ , \tag{12}$$

where $\hat{B}$ is any number $\geq B = \max \sigma(S)$, and by choosing $\varepsilon_{k0} = \varepsilon_{k1} = 1$ and $\delta_{k0} = \delta_{k1} = 1$ in II and IV, respectively. We shall show that in either case II or IV there is quadratic and monotone convergence in these safe modes, no matter how small the frame bound ratio A/B of the initial frame (g, a, b) is as long as it is positive.

The smart but risky modes are obtained by leaving g and S as they were and by taking ε_{k0}, ε_{k1} and δ_{k0}, δ_{k1} such that $\varepsilon_{k0}/\varepsilon_{k1}$ and δ_{k0}/δ_{k1} are between A_k and B_k, where A_k and B_k are the best lower and upper frame bound of the current frame operator S_k. We shall consider the choice

$$\varepsilon_{k0} = \|\gamma_k\|^{-1} \ , \qquad \varepsilon_{k1} = \|S_k \gamma_k\|^{-1} \tag{13}$$

and

$$\delta_{k0} = \|\gamma_k\|^{-1} \ , \qquad \delta_{k1} = \|S_k g\|^{-1} \ , \tag{14}$$

that yield $\varepsilon_{k0}/\varepsilon_{k1} \in [A_k, B_k]$ and $\delta_{k0}/\delta_{k1} \in [A_k, B_k]$, indeed. We shall show that for the smart version of II so obtained we have quadratic and monotone convergence of $\gamma_k/\|\gamma_k\|$ to $g^t/\|g^t\|$ when the initial frame bound ratio $A/B > \frac{1}{2}$, and that for the smart version of IV so obtained we have quadratic and monotone convergence of $\gamma_k/\|\gamma_k\|$ to $g^d/\|g^d\|$ when the initial frame bound ratio $A/B > \frac{1}{2}(\sqrt{5}-1)$.

The constants involved in the smart versions of II and IV that describe the convergence speed are considerably better than those involved in the safe modes. An important point is that one can switch from the safe to the smart mode (and vice versa) without changing the limiting window. This gives the opportunity to start in the safe mode, and to change to the smart mode as soon as one is confident that the frame bound ratio A_k/B_k is large enough.

The algorithms with recursion steps III and V are analyzed in detail in [7], especially with respect to convergence behaviour in the smart modes. Also in these cases there are safe but conservative modes, and one can switch freely from safe to smart modes, and vice versa.

It is not the purpose of this contribution to compare the performance of the presented algorithms with existing ones in the literature; we just give an indication of the performance. Preliminary experiments, with the standard Gaussian window $g(t) = 2^{1/4} \exp(-\pi t^2)$ and $a = b = 1/\sqrt{2}$, have shown that the smart versions produce 10^{-15} accurate approximations within 4–7 steps for the algorithms with quadratic convergence and within 2 or 3 steps for the algorithms with cubic convergence. The number of steps required for the safe versions is typically a factor $1\frac{1}{2}$–2 larger.

This contribution is organized as follows. In Sec. 3. we shall present our main tools to analyze the algorithms I–V. These are the spectral mapping theorem to relate the best frame bounds A_{k+1}, B_{k+1} of (γ_{k+1}, a, b) to the best frame bounds A_k, B_k of (γ_k, a, b), the Kantorovich inequality for transferring convergence of frame operators to convergence of the corresponding windows and, finally, some elementary inequalities to produce numbers (like $\varepsilon_{k0}/\varepsilon_{k1}$ and δ_{k0}/δ_{k1}) that are guaranteed to lie between A_k and B_k. In Sec. 4. the effectiveness of these tools for establishing convergence results is demonstrated by redoing the analysis for the algorithm with recursion step I. Next, in Sec. 5., we present a rationale for proposing iterative algorithms of the types II-III and IV–V meant for approximation of g^t and g^d, respectively, with arbitrary order of convergence m. Such a rationale is badly needed since until now nothing has been said about what has led us to write down the recursions II–V with the constants and terms at the right-hand sides that are characteristic for any of these recursions. In Sec. 6. we shall give a detailed analysis of algorithm II using the tools developed in Sec. 3. so as to derive the results for II in safe and smart mode as announced above. In Sec. 7. we shall do a similar effort as in Sec. 6., but now for the algorithm with recursion step IV. Finally, in Sec. 8. we shall announce a number of results from [7]; these include comments on the algorithms III and IV with (conditional) cubic convergence together with an appropriately sharpened version of Kantorovich's inequality.

We conclude this overview by some comments of more historical and/or motivational nature. The algorithm I was proposed by Feichtinger and Strohmer (independently from one another) around 1995 and analyzed, see [6], around 2000. The intuitive idea here was that $S^{-1/2}g = g^t$ is about halfway between g and $S^{-1}g = g^d$ and that this should be reflected in some sense by a recursion approximating it. Here the occurrence of inverses of frame operators should be considered as much less a problem from a numerical point of view than having to form inverse square roots of frame operators. The algorithm IV was proposed to the author by Feichtinger in Vienna, December 2001; apparently, not much can be said about the motivation of it, except that some adequate guess and try work as well as a good feeling for iterative methods has been instrumental here. The algorithms II and the algorithms III, V were proposed by the author himself just before and just after December 2001, respectively. All these inputs lead to the work [7] in which the algorithms II–V were analyzed in their smart modes. The modification of algorithms from smart (but risky) modes to safe modes was prompted by an observation of Hampejs in the spring of 2002, [5], where a scaling of the window g and the frame operator S somewhat similar to (12) and constant ε's and δ's in II and IV, respectively, gave rise to versions of algorithms II and IV converging under less restrictive conditions on A/B than those required in the smart modes considered in [7]. Finally, the author was informed of recent work of M. Lammers [9] in which Newton type methods for the approximation of tight and dual windows for a Gabor frame (g, a, b) were proposed. The approach in [9] depends on a convolution product $\odot_\alpha$ for windows, in which a pre-chosen tight window α occurs, and yields, in general, non-canonical tight and dual windows (without restrictions on the initial frame bounds A, B).

3. Basic tools

In this section we present the basic tools by which we analyze the algorithms presented in Sec. 2.. We concentrate here on algorithms of type I–III to approximate g^t, and we present details for how to use the tools to analyze algorithms of the type IV–V to approximate g^d in Subsec. 5.2..

In the case of algorithms I-III we have recursion steps of the type $\gamma_{k+1} = \varphi_k(S_k)\,\gamma_k$ with φ_k an analytic function. Since γ_k is supposed to approximate g^t and since the frame operator corresponding to (g^t, a, b) is the identity I, the frame bound ratio A_k/B_k of the frame (γ_k, a, b) should approximate 1. The following theorem can be used to relate the frame bounds A_{k+1}, B_{k+1} of $(\gamma_{k+1} = \varphi_k(S_k)\,\gamma_k, a, b)$ to the frame bounds A_k, B_k of (γ_k, a, b); it thus serves as a tool to track the quantity A_k/B_k during the iteration.

Theorem 1: Let (g, a, b) be a Gabor frame with frame operator S and let φ be analytic in an open neighbourhood of $\sigma(S)$. Moreover, assume that φ is positive on $\sigma(S)$. Then $(\varphi(S)\,g, a, b)$ is a Gabor frame with frame operator $S\,\varphi^2(S)$ and best lower and upper frame bounds given by

$$\min_{s\in\sigma(S)} s\,\varphi^2(s) \quad \text{and} \quad \max_{s\in\sigma(S)} s\,\varphi^2(s)\ , \tag{15}$$

respectively. Furthermore,

$$(\varphi(S)\,g)^t = g^t\ , \tag{16}$$

i.e. g and $\varphi(S)\,g$ have the same canonical tight window for the shift parameters a, b.

Proof: For $f \in L^2(\mathbb{R})$ we compute, see the definition (3) of frame operator,

$$\sum_{n,m} (f, (\varphi(S)\, g)_{na,mb})(\varphi(S)\, g)_{na,mb} \; =$$

$$= \; \sum_{n,m} (f, \varphi(S)\, g_{na,mb})\, \varphi(S)\, g_{na,mb} \; =$$

$$= \; \sum_{n,m} (\varphi(S)\, f, g_{na,mb})\, \varphi(S)\, g_{na,mb} \; =$$

$$= \; \varphi(S) \sum_{n,m} (\varphi(S)\, f, g_{na,mb})\, g_{na,mb} \; = \; \varphi(S)\, S\, \varphi(S)\, f \ , \tag{17}$$

showing that the frame operator of $(\varphi(S)\, g, a, b)$ equals $S\, \varphi^2(S)$. In (17) it has been used that S, and hence $\varphi(S)$, commutes with all relevant frame operators and that $\varphi(S)$ is a bounded positive operator of $L^2(\mathbb{R})$. By the spectral mapping theorem, we have that $\sigma(\varphi(S)) = \varphi(\sigma(S))$, and this implies that the best frame bounds of $(\varphi(S)\, g, a, b)$ are given by the numbers in (15).

As to (16) we observe that $S\, \varphi^2(S)$ is positive and that, see (5),

$$(\varphi(S)\, g)^t = (S\, \varphi^2(S))^{-1/2}\, \varphi(S)\, g = S^{-1/2} g = g^t \ . \tag{18}$$

This completes the proof. □

Note: The first half of Theorem 1 is also true when φ is just real, rather than positive, on $\sigma(S)$, except that $(\varphi(S)g, a, b)$ does not need to be a frame.

For the iterations of the type $\gamma_{k+1} = \varphi_k(S_k)\, \gamma_k$ that we consider in algorithms I-III there is the following consequence of Theorem 1 and the fact that $\sigma(S_k) \subset [A_k, B_k]$.

Corollary 2: *Assume that g is as in Theorem 1 and that $(\gamma_k)_{k=0,1,\ldots}$ is a sequence in $L^2(\mathbb{R})$ generated as $\gamma_0 = g$, $\gamma_{k+1} = \varphi_k(S_k)\, \gamma_k$, $k = 0, 1, \ldots$, where S_k is the frame operator corresponding to (γ_k, a, b) and φ_k is analytic around and positive on $\sigma(S_k)$. Then*

$$A_{k+1} = \min_{s \in \sigma(S_k)} \; s\, \varphi_k^2(s) \geq \min_{s \in [A_k, B_k]} \; s\, \varphi_k^2(s) \ , \tag{19}$$

$$B_{k+1} = \max_{s \in \sigma(S_k)} \; s\, \varphi_k^2(s) \leq \max_{s \in [A_k, B_k]} \; s\, \varphi_k^2(s) \ , \tag{20}$$

and

$$\gamma_k^t = S_k^{-1/2}\gamma_k = S^{-1/2}g = g^t \tag{21}$$

for all $k = 0, 1, \ldots$.

With Corollary 2 in hand, we can monitor the frame bound ratio $Q_k = A_k/B_k$. It can be expected that γ_k is close to g^t when A_k/B_k is close to 1. This can be made more explicit by using the following inequalities. Let T be a positive linear operator of a Hilbert space H and let $C = \min\sigma(T) > 0$, $D = \max\sigma(T) < \infty$. Then there holds

$$\frac{2CD}{C^2+D^2} \le \frac{\|Tf\|^2}{\|f\|\,\|T^2f\|} \le 1\ , \qquad f \in H\ ; \tag{22}$$

the first inequality here is due to Kantorovich, see [8].

Theorem 3: Assume that γ_k is as in Corollary 2. Then

$$\Big\|\frac{\gamma_k}{\|\gamma_k\|} - \frac{g^t}{\|g^t\|}\Big\| \le (1 - Q_k^{1/4})\sqrt{\frac{2}{1+Q_k^{1/2}}}\ , \tag{23}$$

where $Q_k = A_k/B_k$.

Proof: We have by (21) that $g^t = S_k^{-1/2}\gamma_k$. Hence

$$\begin{aligned}\Big\|\frac{\gamma_k}{\|\gamma_k\|} - \frac{g^t}{\|g^t\|}\Big\|^2 &= 2\Big(1 - \mathrm{Re}\Big[\frac{(\gamma_k, g^t)}{\|\gamma_k\|\,\|g^t\|}\Big]\Big) \ = \\ &= 2\Big(1 - \frac{\|S_k^{-1/4}\gamma_k\|^2}{\|\gamma_k\|\,\|S_k^{-1/2}\gamma_k\|}\Big) \ \le \\ &\le 2\Big(1 - \frac{2B_k^{-1/4}A_k^{-1/4}}{B_k^{-1/2}+A_k^{-1/2}}\Big) \ = \\ &= 2\,\frac{(B_k^{1/4}-A_k^{1/4})^2}{A_k^{1/2}+B_k^{1/2}} = 2\,\frac{(1-Q_k^{1/4})^2}{1+Q_k^{1/2}}\ . \end{aligned} \tag{24}$$

Here we have used the first inequality in (22) with $T = S_k^{-1/2}$ and $C = B_k^{-1/4}$, $D = A_k^{-1/4}$ while choosing $f = \gamma_k$. This completes the proof. □

Note: The monotonicity statements, as occur in the formulation of the results in Sec. 2., refer to the monotonicity of the Q_k and not to the monotonicity of the quantities at the left-hand side of (23).

In the smart versions of the algorithms the availability of numbers between the best frame bounds A_k and B_k, without knowing A_k and B_k themselves, is required. For this we have the following result.

Theorem 4: Let (g, a, b) be a Gabor frame with frame operator S and best frame bounds A, B. Then for any $h \in L^2(\mathbb{R})$ there holds

$$A \leq \frac{\|h\|}{\|S^{-1}h\|} \leq \frac{\|Sh\|}{\|h\|} \leq B \ . \tag{25}$$

Furthermore, we have

$$A \leq \frac{\|g\|}{\|S^{-1}g\|} \leq \frac{\|g\|^2}{(S^{-1}g, g)} = \frac{1}{ab}\,\|g\|^2 \leq \frac{(Sg, g)}{\|g\|^2} \leq \frac{\|Sg\|}{\|g\|} \leq B \ . \tag{26}$$

Proof: These inequalities are elementary, and the identity

$$\frac{\|g\|^2}{(S^{-1}g, g)} = \frac{1}{ab}\,\|g\|^2 \tag{27}$$

in the middle of the chain in (26) follows from $(g, g^d) = (g, S^{-1}g) = ab$ (Wexler-Raz). This completes the proof. □

4. Analysis of recursion I to approximate g^t

In this section we redo the analysis given in [6], Sec. 4 to demonstrate the effectiveness of the tools developed in Sec. 3. to analyze the algorithm with I as recursion step. Hence we consider the recursion

$$\gamma_0 = g \ ; \qquad \gamma_{k+1} = \tfrac{1}{2}\,\alpha_k \gamma_k + \tfrac{1}{2}\,\beta_k S_k^{-1} \gamma_k \ , \quad k = 0, 1, \dots \ , \tag{28}$$

where $\alpha_k > 0$, $\beta_k > 0$, and we pay special attention to the case that

$$\alpha_k = \|\gamma_k\|^{-1} \ , \qquad \beta_k = \|S_k^{-1}\gamma_k\|^{-1} \ . \tag{29}$$

We have in the present case that

$$\gamma_{k+1} = \varphi_k(S_k)\,\gamma_k \ ; \qquad \varphi_k(s) = \tfrac{1}{2}\,\alpha_k + \tfrac{1}{2}\,\beta_k s^{-1} \ , \tag{30}$$

where we observe that φ_k is analytic and positive on $(0, \infty)$. Hence by Corollary 2 there holds

$$A_{k+1} \geq \min_{s \in [A_k, B_k]} s\,\varphi_k^2(s) \ , \qquad B_{k+1} \leq \max_{s \in [A_k, B_k]} s\,\varphi_k^2(s) \ , \tag{31}$$

and thus

$$Q_{k+1} = \frac{A_{k+1}}{B_{k+1}} \geq \frac{\min\, s\,\varphi_k^2(s)}{\max\, s\,\varphi_k^2(s)} \ , \tag{32}$$

where min and max are taken over $[A_k, B_k]$.

In Fig. 1 we have plotted for a fixed value of $\alpha > 0$, $\beta > 0$ the graph of the mapping

$$s > 0 \to s\,\varphi^2(s) \ ; \qquad \varphi(s) = \tfrac{1}{2}\,\alpha + \tfrac{1}{2}\,\beta\, s^{-1} \ , \tag{33}$$

and we consider the minimum and maximum of $s\,\varphi^2(s)$ over an interval $[A, B]$ where we distinguish between the cases

$$\text{(i) } A \le \frac{\beta}{\alpha} \le B \ , \qquad \text{(ii) } A \le B < \frac{\beta}{\alpha} \ , \qquad \text{(iii) } \frac{\beta}{\alpha} < A \le B \ . \tag{34}$$

It is thus seen that

$$\text{Case (i)} : \quad \frac{\text{min over } [A,B]}{\text{max over } [A,B]} = \frac{\alpha\beta}{\text{max over } \{A,B\}} \ , \tag{35}$$

$$\text{Case (ii)} : \quad \frac{\text{min over } [A,B]}{\text{max over } [A,B]} = \frac{B\,\varphi^2(B)}{A\,\varphi^2(A)} \ , \tag{36}$$

$$\text{Case(iii)} : \quad \frac{\text{min over } [A,B]}{\text{max over } [A,B]} = \frac{A\,\varphi^2(A)}{B\,\varphi^2(B)} \ . \tag{37}$$

An elementary calculation shows that the three right-hand sides of (35)–(37) are all $> A/B$.

Returning to (32) we see that $Q_{k+1} > Q_k$, no matter what A_k and B_k are as long as $A_k > 0$, $B_k < \infty$. Hence, in each step of the recursion in (28) the frame (γ_k, a, b) gets tighter. Furthermore, this process of tightening up is seen to be fastest when we consistently choose α_k, β_k such that case (i) in (34) occurs with α_k, β_k, A_k, B_k instead of α, β, A, B. In the latter case it can be shown by elementary means that

$$\frac{4Q_k}{(1+Q_k)^2} \le \frac{\alpha_k\beta_k}{\max\{A_k\varphi_k^2(A_k), B_k\varphi_k^2(B_k)\}} \le \frac{4Q_k^{1/2}}{(1+Q_k^{1/2})^2} \ . \tag{38}$$

Equality in the left-hand side inequality in (38) occurs if and only if β_k/α_k equals A_k or B_k, and equality in the right-hand side inequality in (38) occurs if and only if $\beta_k/\alpha_k = (A_k B_k)^{1/2}$. Thus we certainly have that

$$Q_{k+1} \ge \frac{4Q_k}{(1+Q_k)^2} = 1 - \Big(\frac{1-Q_k}{1+Q_k}\Big)^2 \ , \tag{39}$$

and this implies at least quadratic convergence of Q_k to 1. Then we can use Theorem 3 in Sec. 3. to transfer convergence of frame operators to windows.

The convergence of Q_k to 1 can be considerably faster than what (39) would give with the equality sign. This is, for instance, the case when $\sigma(S)$

is a small subset of $[A,B]$, so that $\sigma(S_k)$ is a small subset of $[A_k,B_k]$. We note that Q_{k+1} equals the middle quantity in (38) when $\sigma(S)=[A,B]$ (so that $\sigma(S_k)=[A_k,B_k]$). Another instance where a considerable sharpening of (39) occurs is the case that we manage to choose α_k, β_k such that β_k/α_k is close to $(A_kB_k)^{1/2}$ so that near-equality in the second inequality in (38) occurs. Noting that for Q close to 1

$$\Big(\frac{1-Q}{1+Q}\Big)^2 \approx \tfrac{1}{4}\,(1-Q)^2~, \qquad \Big(\frac{1-Q^{1/2}}{1+Q^{1/2}}\Big)^2 \approx \tfrac{1}{16}\,(1-Q)^2~, \tag{40}$$

we see that the convergence constant $\frac{1}{4}$ is improved to $\frac{1}{16}$ when we succeed in choosing β_k/α_k close to $(A_kB_k)^{1/2}$.

The condition that $\beta_k/\alpha_k \in [A_k,B_k]$ can be satisfied in many ways as is evident from Theorem 4. In [6], Sec. 4 the choice (29) has been made, which corresponds to the second term in the chain in (26), so that both members at the right-hand side of the recursion step in (28) have norm $\frac{1}{2}$.

We now point at the fact that there is quadratic convergence in (28), even when one does not pay special attention in choosing α_k, β_k. This is the case, for instance, when one chooses $\alpha_k=\beta_k=1$ for all k. Consider this case in the event that case (iii) in (37) holds. Then there holds, see Fig. 1 (iii),

$$A_{k+1}=1+\frac{(A_k-1)^2}{4A_k}~, \qquad B_{k+1}=1+\frac{(B_k-1)^2}{4B_k}~, \tag{41}$$

and from this one sees that both A_k and B_k tend to 1 quadratically (equality in (41) since the best frame bounds A_k, B_k are in $\sigma(S_k)$). A similar thing occurs when case (i) or (ii) holds in (34). Evidently, this careless strategy comes at a price when A, B are both very small or very large. And also the asymptotic convergence constant is normally not better than $\frac{1}{4}$, where a potential $\frac{1}{16}$ could be obtained when one is successful in choosing β_k/α_k somewhere in the middle of the interval $[A_k,B_k]$.

5. Proposing iterations without inversions

In this section we develop a rationale for proposing iterative algorithms for the approximation of g^t or g^d using no frame operator inversions and with a given order m of convergence. This yields the algorithms II–III with

$$\varepsilon_{k0}=\|\gamma_k\|^{-1}~, \qquad \varepsilon_{k1}=\|S_k\gamma_k\|^{-1}~, \qquad \varepsilon_{k2}=\|S_k^2\gamma_k\|~, \tag{42}$$

and algorithms IV–V with

$$\delta_{k0}=\|\gamma_k\|^{-1}~, \qquad \delta_{k1}=\|S_kg\|^{-1}~, \qquad \delta_{k2}=\|SS_k\gamma_k\|^{-1}~, \tag{43}$$

respectively, corresponding to the cases $m = 2$ and 3. In Subsec. 5.1. we consider this rationale for iterations meant to approximate g^t. In Subsec. 5.2. we consider this issue for the approximation of g^d, and there we first show how the tools developed in Sec. 3. are to be used for analyzing iterations for g^d.

5.1. *Iterations for g^t*

Let $m = 2, 3, \ldots$. We consider an iteration step

$$\gamma_{k+1} = \varphi_k(S_k)\,\gamma_k \ , \tag{44}$$

with φ_k to be chosen such that the following holds:

$$\exists_{c>0}[S_k \approx cI] \Rightarrow \|S_{k+1} - cI\| = O(\|S_k - cI\|^m) \ . \tag{45}$$

The condition (45) means that the asymptotic order of convergence of the iteration on the level of frame operators is equal to m.

So let $c > 0$ be such that $S_k \approx cI$. Note that

$$S_{k+1} = S_k \varphi_k^2(S_k) \tag{46}$$

by Theorem 1 in Sec. 3.. Assuming the operators in (46) to be positive, so that square roots of them may be taken, we require that

$$S_k^{1/2}\varphi_k(S_k) = c^{1/2} I + O(\|S_k - cI\|^m) \ . \tag{47}$$

Up to errors $O(\|S_k - cI\|^m)$ we calculate

$$\begin{aligned} S_k^{-1/2} &= c^{-1/2}(I - (I - c^{-1}S_k))^{-1/2} \ = \\ &= c^{-1/2} \sum_{i=0}^{m-1} (-1)^i \binom{-1/2}{i} (I - c^{-1}S_k)^i \ = \\ &= c^{-1/2} \sum_{j=0}^{m-1} a_{mj}\, c^{-j}\, S_k^j \ . \end{aligned} \tag{48}$$

The quantities a_{mj} in (48) are given explicitly as

$$a_{mj} = \sum_{i=j}^{m-1} (-1)^{i+j} \binom{-1/2}{i} \binom{i}{i-j} \ , \qquad j = 0, \ldots, m-1 \ , \tag{49}$$

as can be obtained by binomial expansion of each of the terms $(I - c^{-1}S_k)^i$ in (48). This then suggests to take

$$\varphi_k(s) = \sum_{j=0}^{m-1} a_{mj}\, c^{-j}\, s^j \ . \tag{50}$$

The quantities c^{-j} are not known, however, and should be estimated one way or another. One can use, for instance

$$\frac{||\gamma_k||}{||S_k^j\gamma_k||}\ , \qquad j=0,1,...,m-1\ , \tag{51}$$

as an estimate for c^{-j}. Dividing through by the overall constant $||\gamma_k||$, we finally get the recursion step

$$\gamma_{k+1}=\sum_{j=0}^{m-1} a_{mj}\,\frac{S_k^j\gamma_k}{||S_k^j\gamma_k||}\ , \tag{52}$$

where the a_{mj} are given explicitly by (49). We may note here that $\sum_{j=0}^{m-1} a_{mj}=1$ and that all terms $S_k^j\gamma_k/||S_k^j\gamma_k||$ at the right-hand side of (52) have unit norm.

We point out that we have argued heuristically here, our purpose only being to come up with a sound proposal for an iterative methode with order of convergence m. In particular, the matter of taking square roots in (47) deserves special care, and actually nothing has been proved yet. In Sec. 6. and in [7], Sec. 8, these matters are take care of in detail for the cases $m=2$ and 3, respectively.

Examples: For $m=2$ and $m=3$ one gets the recursion steps

$$\gamma_{k+1}=\frac{3}{2}\,\frac{\gamma_k}{||\gamma_k||}-\frac{1}{2}\,\frac{S_k\gamma_k}{||S_k\gamma_k||} \tag{53}$$

and

$$\gamma_{k+1}=\frac{15}{8}\,\frac{\gamma_k}{||\gamma_k||}-\frac{5}{4}\,\frac{S_k\gamma_k}{||S_k\gamma_k||}+\frac{3}{8}\,\frac{S_k^2\gamma_k}{||S_k^2\gamma_k||}\ , \tag{54}$$

respectively.

5.2. *Iterations for g^d*

We next consider the problem of proposing m^{th} order convergent recursions for the approximation of g^d. To that end we consider algorithms of the type

$$\gamma_0=g\,;\quad \gamma_{k+1}=\psi_k(S,S_k)\,\gamma_k\ , \qquad k=0,1,...\ , \tag{55}$$

where S and S_k are the frame operators corresponding to (g,a,b) and (γ_k,a,b), respectively, and ψ_k is an analytic function of two variables. We first need to develop tools as in Sec. 3. for the present situation.

By induction it is seen from (55) that there are analytic functions f_k, g_k such that

$$\gamma_k = f_k(S)\, g \ , \qquad S_k = g_k(S) \ , \tag{56}$$

where $g_k(s) = s\, f_k^2(s)$, see Theorem 1 in Sec. 3.. Assuming all operators to be positive it follows that

$$\gamma_k = S^{-1/2}\, S_k^{1/2}\, g \ , \qquad g^d = S^{-1} g = (SS_k)^{-1/2}\, \gamma_k \ . \tag{57}$$

For later use we also note that we have

$$(SS_k)^{1/2}\, \gamma_k = S_k\, g \ . \tag{58}$$

From the second item in (57) we see that

$$\Big\| \frac{\gamma_k}{\|\gamma_k\|} - \frac{g^d}{\|g^d\|} \Big\| = \Big\| \frac{\gamma_k}{\|\gamma_k\|} - \frac{(SS_k)^{-1/2}\, \gamma_k}{\|(SS_k)^{-1/2}\, \gamma_k\|} \Big\| \ , \tag{59}$$

and this shows that we are in a situation completely similar to what we have encountered in Sec. 3. for g^t. Accordingly, we should concentrate on convergence of $SS_k/\|SS_k\|$ to I.

We shall first show that $S_{k+1} = S_k \psi^2(S, S_k)$. Indeed, we have from (56) that

$$\gamma_{k+1} = \psi_k(S, g_k(S))\, f_k(S)\, g \ , \tag{60}$$

so that by Theorem 1 in Sec. 3.

$$S_{k+1} = S[\psi_k(S, g_k(S))\, f_k(S)]^2 = S f_k^2(S)\, \psi_k^2(S, g_k(S)) = S_k \psi_k^2(S, S_k) \ . \tag{61}$$

We next want ψ_k to satisfy the following:

$$\exists_{c>0}\, [SS_k \approx cI] \Rightarrow \|SS_{k+1} - cI\| = O(\|SS_k - cI\|^m) \ . \tag{62}$$

Here $m = 2, 3, \ldots$ is the envisaged convergence order of the method. Again assuming that all square roots may be taken, we let

$$Z_k = (SS_k)^{1/2} \ . \tag{63}$$

Then there holds $Z_{k+1} = Z_k \psi_k(S, S_k)$ by the above. Now assume that $c > 0$ is such that $Z_k \approx c^{1/2} I$. We require thus that

$$Z_k \psi_k(S, S_k) = c^{1/2} I + O(\|Z_k - c^{1/2} I\|^m) \ . \tag{64}$$

Up to errors $O(\|Z_k - c^{1/2} I\|^m)$ there holds, compare (48)

$$Z_k^{-1} = c^{-1/2}(I - (I - c^{-1/2} Z_k))^{-1} = c^{-1/2} \sum_{j=0}^{m-1} b_{mj}\, c^{-1/2j}\, Z_k^j \ , \tag{65}$$

where the b_{mj} have the explicit form

$$b_{mj} = (-1)^j \sum_{i=j}^{m-1} \binom{i}{i-j} , \qquad j = 0, ..., m-1 . \tag{66}$$

This suggests to take

$$\psi_k(S, S_k) = \sum_{j=0}^{m-1} b_{mj} \, c^{-j/2} \, Z_k^j , \tag{67}$$

so that the proposed recursion step becomes

$$\gamma_{k+1} = \sum_{j=0}^{m-1} b_{mj} \, c^{-j/2} \, Z_k^j \, \gamma_k . \tag{68}$$

There are now two problems with the proposal (68) that should be addressed. First, the numbers $c^{-j/2}$ at the right-hand side of (68) are unknown. We estimate these $c^{-j/2}$ as

$$\frac{\|\gamma_k\|}{\|Z_k^j \gamma_k\|} , \qquad j = 0, 1, ..., m-1 . \tag{69}$$

Secondly, we should find a way to compute $Z_k^j \gamma_k$ without having to take square roots of operators, see (63). For this we have (58) and accordingly for $i = 0, 1, ...$

$$Z_k^{2i} \gamma_k = (S S_k)^i \gamma_k \; ; \qquad Z_k^{2i+1} \gamma_k = (S S_k)^i \, S_k \, g . \tag{70}$$

Finally, dividing through by $\|\gamma_k\|$, we obtain as proposed recursion step

$$\gamma_{k+1} = \sum_{j=0}^{m-1} b_{mj} \, \frac{Z_k^j \gamma_k}{\|Z_k^j \gamma_k\|} , \tag{71}$$

where the b_{mj} are given in (66) and the $Z_k^j \gamma_k$ are to be computed according to (70). We note here that $\sum_{j=0}^{m-1} b_{mj} = 1$ and that all terms $Z_k^j \gamma_k / \|Z_k^j \gamma_k\|$ at the right-hand side of (71) have unit norm.

For use in Sec. 7. we mention the following result. When we have S, S_k, Z_k as above and $\gamma_{k+1} = \varphi_k(Z_k) \, \gamma_k$, with φ_k analytic around and positive on $\sigma(Z_k)$, then there holds

$$Z_{k+1} = Z_k \, \varphi_k(Z_k) . \tag{72}$$

Indeed, with (56) we see that

$$\gamma_{k+1} = \varphi_k((S \, g_k(S))^{1/2}) \, f_k(S) \, g \tag{73}$$

so that by Theorem 1 in Sec. 3.

$$S_{k+1} = S\,[\varphi_k((S\,g_k(S))^{1/2})\,f_k(S)]^2 = S\,f_k^2(S)\,\varphi_k^2(Z_k) = S_k \varpi_k^2(Z_k)\ . \quad (74)$$

Then multiplying by S and taking square roots in the far left- and the far right-hand side of (74) we get (72). In particular, we see in the case of (71) that

$$Z_{k+1} = \sum_{j=0}^{m-1} \frac{b_{mj}}{\|Z_k^j \gamma_k\|}\, Z_k^j\ . \quad (75)$$

As in Subsec. 5.1. we must point out that we haven't proved much in this subsection, our aim just being to develop a mechanism for soundly proposing recursion steps for the m^{th} order convergent methods.

Examples: For $m = 2$ and 3 one gets the recursion steps

$$\gamma_{k+1} = 2\,\frac{\gamma_k}{\|\gamma_k\|} - \frac{S_k g}{\|S_k g\|} \quad (76)$$

and

$$\gamma_{k+1} = 3\,\frac{\gamma_k}{\|\gamma_k\|} - 3\,\frac{S_k g}{\|S_k g\|} + \frac{S\,S_k\,\gamma_k}{\|S\,S_k\,\gamma_k\|}\ , \quad (77)$$

respectively.

6. Analysis of recursion II to approximate g^t

In this section we consider the recursion

$$\gamma_0 = g\,;\quad \gamma_{k+1} = \tfrac{3}{2}\,\varepsilon_{k0}\gamma_k - \tfrac{1}{2}\,\varepsilon_{k1}\,S_k\,\gamma_k\ ,\qquad k = 0, 1, \ldots\ , \quad (78)$$

with $\varepsilon_{k0} > 0$, $\varepsilon_{k1} > 0$, and we pay special attention to the case that

$$\varepsilon_{k0} = \|\gamma_k\|^{-1}\ ,\qquad \varepsilon_{k1} = \|S_k\gamma_k\|^{-1}\ . \quad (79)$$

There holds now

$$\gamma_{k+1} = \varphi_k(S_k)\,\gamma_k\ ;\qquad \varphi_k(s) = \tfrac{3}{2}\,\varepsilon_{k0} - \tfrac{1}{2}\,\varepsilon_{k1} s\ . \quad (80)$$

In order to apply Theorem 1 in Sec. 3. we need that $\varphi_k(s) > 0$, $s \in \sigma(S_k)$, and so we assume

$$\frac{\varepsilon_{k0}}{\varepsilon_{k1}} > \tfrac{1}{3}\,B_k\ . \quad (81)$$

We can now safely pass to square roots of frame operators, whence we let for $k = 0, 1, \ldots$

$$Z_k = S_k^{1/2}\ ;\qquad E_k = A_k^{1/2} = \min \sigma(Z_k)\ ,\quad F_k = B_k^{1/2} = \max \sigma(Z_k)\ . \quad (82)$$

Then there holds

$$Z_{k+1} = (S_k \varphi_k^2(S_k))^{1/2} = Z_k(\tfrac{3}{2}\,\varepsilon_{k0} I - \tfrac{1}{2}\,\varepsilon_{k1} Z_k^2)\ . \tag{83}$$

Since $\sigma(Z_k) \subset [E_k, F_k]$, we have by the spectral mapping theorem in a similar fashion as in (31)–(32) that

$$\frac{E_{k+1}}{F_{k+1}} \geq \frac{\min z(\frac{3}{2}\,\varepsilon_{k0} - \frac{1}{2}\,\varepsilon_{k1} z^2)}{\max z(\frac{3}{2}\,\varepsilon_{k0} - \frac{1}{2}\,\varepsilon_{k1} z^2)}\ , \tag{84}$$

where the min and max are taken over $[E_k, F_k]$.

In Fig. 2 we have plotted for fixed values of $\varepsilon_0 > 0$, $\varepsilon_1 > 0$ and of $E > 0$, $F > 0$ such that $\varepsilon_0/\varepsilon_1 > \frac{1}{3} F^2$ (see (81)) and $E < F$ the graph of the mapping

$$z > 0 \to z\,\varphi(z^2)\ ; \qquad \varphi(s) = \tfrac{3}{2}\,\varepsilon_0 - \tfrac{1}{2}\,\varepsilon_1 s, \tag{85}$$

and we consider the minimum and maximum of $z\,\varphi(z^2)$ over the interval $[E, F]$, where we distinguish between the cases

$$\text{(i) } E \leq \left(\frac{\varepsilon_0}{\varepsilon_1}\right)^{1/2} \leq F\ , \quad \text{(ii) } E \leq F < \left(\frac{\varepsilon_0}{\varepsilon_1}\right)^{1/2}, \quad \text{(iii) } \left(\frac{\varepsilon_0}{\varepsilon_1}\right)^{1/2} < E \leq F\ . \tag{86}$$

It is thus seen that

$$\text{Case (i)}: \quad \frac{\text{min over } [E,F]}{\text{max over } [E,F]} = \frac{\text{value at } (\varepsilon_0/\varepsilon_1)^{1/2}}{\text{max over } \{E,F\}}\ , \tag{87}$$

$$\text{Case (ii)}: \quad \frac{\text{min over } [E,F]}{\text{max over } [E,F]} = \frac{\text{value at } E}{\text{value at } F}\ , \tag{88}$$

$$\text{Case (iii)}: \quad \frac{\text{min over } [E,F]}{\text{max over } [E,F]} = \frac{\text{value at } F}{\text{value at } E}\ , \tag{89}$$

where the minima, maxima and values refer to the function $z\,\varphi(z^2)$ in (85). Note that the condition $\varepsilon_0/\varepsilon_1 > \frac{1}{3} F^2$ ensures that $z\,\varphi(z^2) > 0$ for $z \in [E, F]$.

Returning to (84), an elementary analysis (detailed in [7], Sec. 4) shows that in case (i), with ε_{0k}, ε_{1k}, E_k, F_k instead of ε_0, ε_1, E, F, there holds

$$\frac{E_{k+1}}{F_{k+1}} \geq \frac{F_k}{E_k}\left(\frac{3}{2} - \frac{1}{2}\left(\frac{F_k}{E_k}\right)^2\right)\ , \tag{90}$$

with equality when $E_k = (\varepsilon_{0k}/\varepsilon_{1k})^{1/2}$. The right-hand side of (87) is increasing in $E_k/F_k \in (0, 1]$ and exceeds E_k/F_k if and only if $E_k/F_k > \frac{1}{2}\sqrt{2}$.

Hence in case (i) there holds

$$\frac{E_k}{F_k} > \tfrac{1}{2}\sqrt{2} \Rightarrow \frac{E_{k+1}}{F_{k+1}} > \frac{E_k}{F_k} \ . \tag{91}$$

A further elementary analysis shows that, in case (i), the maximum value of the right-hand side of (86) equals

$$\tfrac{1}{2}\,(E_k F_k^2 + E_k^2 F_k)/(\tfrac{1}{3}\,(E_k^2 + E_k F_k + F_k^2))^{3/2} \ , \tag{92}$$

and occurs when

$$\frac{\varepsilon_{0k}}{\varepsilon_{1k}} = \tfrac{1}{3}\,(E_k^2 + E_k F_k + F_k^2) \ . \tag{93}$$

In terms of the frame operators $S_k = Z_k^2$ and their frame bounds $A_k = E_k^2$, $B_k = F_k^2$ with frame bound quotients $Q_k = A_k/B_k$ the results for case (i) can be summarized as follows. We are in case (i) if and only if

$$\frac{\varepsilon_{0k}}{\varepsilon_{1k}} \in [A_k, B_k] \ , \tag{94}$$

and this condition can be satisfied in various ways, see Theorem 4 in Sec. 3., for instance by taking ε_{0k}, ε_{1k} as in (79). Also, in case (i), there holds

$$Q_{k+1} \geq Q_k^{-1}(\tfrac{3}{2} - \tfrac{1}{2}\,Q_k^{-1}) = 1 - \frac{Q_k - \frac{1}{4}}{Q_k^2}\,(1 - Q_k)^2 \ , \tag{95}$$

and the right-hand side of (95) exceeds Q_k when $Q_k > \frac{1}{2}$. It is thus seen that $Q_k \to 1$, at least quadratically and monotonically, when $Q_0 = A/B > \frac{1}{2}$, where A and B are the best frame bounds of (g, a, b). Finally, still in case (i), when $\sigma(S) = [A, B]$ so that $\sigma(S_k) = [A_k, B_k]$ for all $k = 0, 1, \ldots$, we have from (92) that

$$Q_{k+1} \leq \frac{27}{4}\ \frac{(Q_k^{1/2} + Q_k)^2}{(1 + Q_k^{1/2} + Q_k)^3} \ =$$

$$= 1 - \frac{(2 + Q_k^{1/2})^2\,(\frac{1}{2} + Q_k^{1/2})^2}{(1 + Q_k^{1/2})^2\,(1 + Q_k^{1/2} + Q_k)^3}\,(1 - Q_k)^2 \ , \tag{96}$$

with equality if and only if

$$\frac{\varepsilon_{0k}}{\varepsilon_{1k}} = \tfrac{1}{3}\,(A_k + (A_k B_k)^{1/2} + B_k) \ . \tag{97}$$

Hence, the worst-case tightening up result as given in (95) has a right-hand side $\approx\ 1 - \frac{3}{4}\,(1 - Q_k)^2$ and the best-case tightening up result in (96) has a right-hand side $\approx\ 1 - \frac{3}{16}\,(1 - Q_k)^2$ when Q_k is close to 1. This latter case is often more realistic, especially when we manage to take $\varepsilon_{0k}/\varepsilon_{1k}$ well in the middle of the interval $[A_k, B_k]$. We may also observe that the quantity at the right-hand side of (96) exceeds Q_k, no matter how small Q_k is. We conclude from this discussion that the sufficient condition $Q_0 = A/B > \frac{1}{2}$ for monotone and quadratic convergence with the choice ε_{0k}, ε_{1k} as in (79) is quite often much too stringent.

We now discuss the other two cases in (86) and we start with case (ii). It follows from an elementary analysis, or an inspection of the graph of $z\,\varphi(z^2)$, see Fig. 2(ii) and (85), that in case (ii) one has

$$\frac{E_{k+1}}{F_{k+1}} > \frac{E_k}{F_k} \ , \tag{98}$$

no matter how small E_k/F_k is. Hence this is a safe mode for the iteration (78) in which in each step the frame gets tighter. One can get in this safe mode as follows. Replace

$$g \text{ by } g/\hat{B}^{1/2} \ , \qquad S \text{ by } S/\hat{B} \ , \tag{99}$$

where $\hat{B}$ is any number $\geq B$ (the best upper frame bound of (g, a, b)), and take $\varepsilon_{k0} = \varepsilon_{k1} = 1$. It is easy to see that, directly in terms of S_k and A_k, B_k, there holds

$$A_{k+1} = A_k(\tfrac{3}{2} - \tfrac{1}{2}\,A_k^2) = 1 - \tfrac{1}{2}\,(A_k + 2)(1 - A_k)^2 \ , \tag{100}$$

$$B_{k+1} = B_k(\tfrac{3}{2} - \tfrac{1}{2}\,B_k^2) = 1 - \tfrac{1}{2}\,(B_k + 2)(1 - B_k)^2 \ , \tag{101}$$

(equality here since $A_k, B_k \in \sigma(S_k)$ being best frame bounds). From this it follows that both A_k and B_k tend to 1, monotonically and quadratically. Hence, in this safe mode (ii), there is quadratic convergence no matter how small the frame bound ratio A/B of the initial frame (g, a, b) is. Of course, the constants governing the convergence in the safe mode are not as good as those in the smart but risky mode (i). And also, when both A and B are very small compared to 1, it may take a large number of iterations before actual (quadratic) convergence takes place.

Case (ii) in (86) is neither smart nor safe and should therefore be avoided at all times.

We conclude this section by pointing at the opportunity of switching between the safe mode (ii and the smart mode (i): we have $\gamma_k^t = g^t$, no matter how we choose ε_{k0}, ε_{k1} and whether we prescale g and S as in (99) or not. This means that one can combine the advantages of both modes by first scaling g and S as in (99) and iterating with $\varepsilon_{k0} = \varepsilon_{k1} = 1$ until one is confident that $A_k/B_k > \frac{1}{2}$ and then switching to mode (i) with ε_{k0}, ε_{k1} as in (79), for instance.

7. Analysis of recursion IV to approximate g^d

In this section we consider the recursion

$$\gamma_0 = g \ ; \qquad \gamma_{k+1} = 2\delta_{k0}\gamma_k - \delta_{k1}\, S_k\, g\,, \quad k = 0, 1, \ldots\,, \tag{102}$$

with $\delta_{k0} > 0$, $\delta_{k1} > 0$, and we pay special attention to the case that

$$\delta_{k0} = \|\gamma_k\|^{-1}\ , \qquad \delta_{k1} = \|S_k\, g\|^{-1}\ . \tag{103}$$

We follow the plan of the analysis of recursion II as given in Sec. 6. rather closely.

We have, see for instance (61), (72)

$$S_{k+1} = S_k(2\delta_{k0}I - \delta_{k1}(SS_k)^{1/2})^2\ , \tag{104}$$

provided that S_k is a positive operator so that the square root at the right-hand side can be taken. Hence, with

$$Z_k = (SS_k)^{1/2}\ ; \qquad E_k = \min\sigma(Z_k)\,, \quad F_k = \max\sigma(Z_k)\ , \tag{105}$$

we assume that

$$\frac{\delta_{k0}}{\delta_{k1}} > \tfrac{1}{2}\, F_k\ , \tag{106}$$

so that the operator at the right-hand side of (104) is positive indeed. Multiplying either side of (104) by S and taking square roots, we get (also see (72))

$$Z_{k+1} = Z_k(2\delta_{k0}I - \delta_{k1}Z_k)\ . \tag{107}$$

We are now practically in a similar position as in Sec. 6., (82)–(84). Accordingly, we get

$$\frac{E_{k+1}}{F_{k+1}} \geq \frac{\min z(2\delta_{k0} - \delta_{k1}z)}{\max z(2\delta_{k0} - \delta_{k1}z)}\ , \tag{108}$$

where min and max are taken over $[E_k, F_k]$, and three cases

$$\text{(i) } E_k \leq \frac{\delta_{0k}}{\delta_{1k}} \leq F_k\ , \qquad \text{(ii) } E_k \leq F_k < \frac{\delta_{0k}}{\delta_{1k}}\ , \qquad \text{(iii) } \frac{\delta_{0k}}{\delta_{1k}} < E_k \leq F_k \tag{109}$$

have to be considered. Here case (i) corresponds to the smart but risky mode, case (ii) corresponds to the safe but conservative mode, while case (iii) is the mode that is neither smart nor safe.

The results are then as follows. Let $Q_k = E_k/F_k$. In case (i) in (109) we have

$$Q_{k+1} \geq Q_k^{-1}(2 - Q_k^{-1}) = 1 - \Big(\frac{1 - Q_k}{Q_k}\Big)^2\ . \tag{110}$$

The right-hand side of (110) is increasing in $Q_k \in (0,1]$ and exceeds Q_k when $Q_k > \frac{1}{2}(\sqrt{5}-1)$. Therefore we have that $Q_k \to 1$, monotonically and at least quadratically, when $Q_0 = E_0/F_0 = A/B > \frac{1}{2}(\sqrt{5}-1)$, provided that we are in case (i) during the iteration process. This latter condition can be satisfied in many ways, see Theorem 4 in Sec. 3., for instance, by taking δ_{k0}, δ_{k1} as in (103). Indeed, for this choice of δ_{k0}, δ_{k1}) we have, see (58),

$$\frac{\delta_{0k}}{\delta_{1k}} = \frac{||S_k g||}{||\gamma_k||} = \frac{||Z_k \gamma_k||}{||\gamma_k||} \in [E_k, F_k] \ . \tag{111}$$

As a best-case result in case (i) we have the following. Assume that $\sigma(S) = [A, B]$, so that $\sigma(Z_k) = [A_k, B_k]$ for all k. Then there holds

$$Q_{k+1} \leq \frac{4Q_k}{(1+Q_k)^2} = 1 - \Big(\frac{1-Q_k}{1+Q_k}\Big)^2 \ , \tag{112}$$

with equality if and only if

$$\frac{\delta_{0k}}{\delta_{1k}} = \tfrac{1}{2}\,(E_k + F_k) \ . \tag{113}$$

Hence, the worst-case tightening up result (110) can be replaced by the best-case tightening up result (112) when we succeed in choosing δ_{0k}/δ_{1k} right in the middle of $[E_k, F_k]$. Note also that the right-hand side of (112) exceeds Q_k, no matter how small Q_k is.

In the safe mode (ii) in (109) we have

$$\frac{E_{k+1}}{F_{k+1}} > \frac{E_k}{F_k} \ . \tag{114}$$

This safe mode is reached by replacing g and S as in (99) and by taking $\delta_{k0} = \delta_{k1} = 1$. Then we have $0 < E_k \leq F_k <$, and

$$E_{k+1} = E_k(2-E_k) = 1-(1-E_k)^2 \ , \qquad F_{k+1} = F_k(2-F_k) = 1-(1-F_k)^2 \tag{115}$$

(equality here since $E_k, F_k \in \sigma(Z_k)$). Hence in this safe mode there is quadratic convergence, no matter how small the initial frame bound ratio A/B is. However, the convergence constants are not as good as in the smart mode, and when A and B are very small it can take many iterations before actual (quadratic) convergence takes place.

8. Summary of results for iterations III and V

In this section we consider the iterations III and V for which the recursion steps are given by

$$\gamma_{k+1} = \tfrac{15}{8}\,\varepsilon_{k0}\gamma_k - \tfrac{5}{4}\,\varepsilon_{k1}\,S_k\,\gamma_k + \tfrac{3}{8}\,\varepsilon_{k2}\,S_k^2\,\gamma_k \ , \tag{116}$$

and

$$\begin{aligned}\gamma_{k+1} &= 3\delta_{k0}\gamma_k - 3\delta_{k1}\, S_k\, g + \delta_{k2}\, SS_k\, \gamma_k \ = \\ &= 3\delta_{k0}\gamma_k - 3\delta_{k1}\, Z_k\, \gamma_k + \delta_{k2}\, Z_k^2\, \gamma_k \end{aligned} \tag{117}$$

(with $Z_k = (SS_k)^{1/2}$ in (117)), for the approximation of g^t and g^d, respectively. These recursions are considered in all detail in [7], Sec. 8 and 9, with particular attention for smart modes. We summarize the results of [7] here and present some supplements regarding safe modes.

The recursions (116) and (117) are analyzed under conditions that arise when one chooses

$$\varepsilon_{ki} = \|S_k^i \gamma_k\|^{-1} \ \text{ and } \ \delta_{ki} = \|Z_k^i \gamma_k\|^{-1} \ , \tag{118}$$

respectively. In the case of recursion (116), with frame operators S_k of (γ_k, a, b) having frame bounds A_k and B_k, there are the restrictions

$$A_k \le u_k := \frac{\varepsilon_{k1}}{\varepsilon_{k2}} \le B_k \ , \qquad \frac{2A_k B_k}{A_k^2 + B_k^2} \le v_k := \frac{\varepsilon_{k0}\varepsilon_{k2}}{\varepsilon_{k1}^2} \le 1 \ . \tag{119}$$

The two inequalities in (119) correspond to the inequalities

$$C \le \frac{\|T^2 f\|}{\|Tf\|} \le D \ , \qquad \frac{2CD}{C^2 + D^2} \le \frac{\|Tf\|^2}{\|f\|\,\|T^2 f\|} \le 1 \ , \tag{120}$$

valid for a positive linear operator T of a Hilbert space H and $f \in H$, also see (22), with $C = \min \sigma(T)$, $D = \max \sigma(T)$. The restrictions on δ_{ki} in (117) are similar, except that now in (119) we have $u_k = \delta_{k1}/\delta_{k2}$, $v_k = \delta_{k0}\delta_{k2}/\delta_{k1}^2$ and $E_k = \min Z_k$, $F_k = \max Z_k$ instead of A_k, B_k.

Actually, the recursions (116) and (117) are analyzed under a slightly stronger restriction than what the Kantorovich inequality in the second item in (119) gives. This stronger version of the second item in (119) reads

$$\frac{u_k^{-1}\, A_k\, B_k}{(A_k^2 + B_k^2 - u_k^2)^{1/2}} \le v_k \le 1 \ . \tag{121}$$

This sharpening is relevant in the present context in the following sense. In [7] it is shown that for any four numbers A, B, u, t with

$$0 < A \le \frac{AB}{(A^2 + B^2 - u^2)^{1/2}} \le t \le u \le B < \infty \ , \tag{122}$$

there is a Gabor frame $(g, a = 1, b = 1)$ with frame operator S having best frame bounds A, B and such that

$$\frac{\|Sg\|}{\|g\|} = t \ , \qquad \frac{\|S^2 g\|}{\|Sg\|} = u \ . \tag{123}$$

Under condition (119), strengthened according to (121), it is shown in [7] for recursion (116) that

$$\frac{A_k}{B_k} > \frac{3}{7} \Rightarrow \frac{A_{k+1}}{B_{k+1}} > \frac{A_k}{B_k} \ , \tag{124}$$

and that cubic and monotone convergence occurs when $A/B > 3/7$. Under the analogon of condition (119) for recursion (117), strengthened according to (121), it is shown in [7] that

$$\frac{E_k}{F_k} > Q \Rightarrow \frac{E_{k+1}}{F_{k+1}} > \frac{E_k}{F_k} \ , \tag{125}$$

and that cubic and monotone convergence occurs when $A/B > Q$. Here

$$Q = 0.513829766... = \frac{3}{2W^3} - \Big(\Big(\frac{3}{2W^3}\Big)^2 - 1\Big)^{1/2} \tag{126}$$

with W the unique solution in $(0, (3/2)^{1/2})$ of

$$9W^2 - 16W^3 + 9W^4 + 6W^5 + 6W^7 + 2W^9 = 24 \ . \tag{127}$$

No effort is spent in [7] to find the best cases of the recursions (116), (117) under the condition that $\sigma(S) = [A, B]$.

Safe but conservative versions of the recursions (116), (117) can be obtained by replacing g and S as in (99) and by taking

$$\varepsilon_{ki} = 1 \, , \quad \delta_{ki} = 1 \ , \qquad k = 0, 1, \ldots \, , \quad i = 0, 1, 2 \ . \tag{128}$$

In that case there holds

$$0 < A_k \le B_k < 1 \, , \quad 0 < E_k \le F_k < 1 \ , \qquad k = 0, 1, \ldots \ . \tag{129}$$

Furthermore, the quantities $R_k := A_k^{1/2}, B_k^{1/2}$ are recursively given as $R_0 = A^{1/2}, B^{1/2}$ and

$$R_{k+1} = R_k(\tfrac{15}{8} - \tfrac{5}{4}\,R_k^2 + \tfrac{3}{8}\,R_k^4) = 1 - (\tfrac{3}{8}\,R_k^2 + \tfrac{9}{8}\,R_k + 1)(1 - R_k)^3 \ , \qquad k = 0, 1, \ldots \ , \tag{130}$$

and the quantities $U_k := E_k, F_k$ are recursively given as $U_0 = A, B$ and

$$U_{k+1} = U_k(3 - 3U_k + U_k^2) = 1 - (1 - U_k)^3 \ , \qquad k = 0, 1, \ldots \ . \tag{131}$$

It follows then that both A_k, B_k and E_k, F_k converge cubically and monotonically to 1, no matter how small A_k/B_k or E_k/F_k are.

9. Concluding remarks

We have presented three iterative algorithms for the approximation of the canonical tight window $g^t = S^{-1/2}g$ associated with a Gabor frame (g, a, b) with frame operator S and two iterative algorithms for the approximation of the canonical dual window $g^d = S^{-1}g$ associated with (g, a, b). These algorithms require in the k^{th} recursion step the application of the current frame operator S_k (and, in one instance of the algorithms for g^t, of S_k^{-1}) to the current window γ_k and/or to g for algorithms with envisaged quadratic convergence. For the algorithms with envisaged cubic convergence, the operators S_k^2 or SS_k have to be applied, in addition, to γ_k or g. We have developed a number of tools to analyze these algorithms, where a key role is played by the spectral mapping theorem, basic frame operator calculus and basic inequalities in Hilbert space operator theory such as the Kantorovich inequality. We have demonstrated the effectiveness of the developed tools by redoing the analysis in [6] for the approximation of g^t using frame operator inversions. We have presented a rationale for proposing iterative algorithms of the described type with an envisaged convergence order $m = 2, 3, \ldots$, and we have given a detailed analysis of the algorithms for g^t and g^d with quadratic convergence ($m = 2$). Furthermore, we have summarized the results of [7], Secs. 8–9, in which detailed analyses are given of the algorithms with cubic convergence ($m = 3$).

All algorithms we have considered have smart versions, in which excellent convergence behaviour is exhibited conditionally on the condition number of the initial frame operator S, and safe versions, in which there is unconditional convergence of the required order with suboptimal convergence constants. One can freely switch between the safe version and the smart version without altering the limiting window g^t or g^d. Preliminary experiments, with the standard Gaussian window $g(t) = 2^{1/4}\exp(-\pi t^2)$ and $a = b = 1/\sqrt{2}$, have shown that the smart versions produce 10^{-15} accurate approximations within 4–7 steps for the algorithms with quadratic convergence and within 2 or 3 steps for the algorithms with cubic convergence. The number of steps required for the safe versions is typically a factor $1\frac{1}{2}$–2 larger.

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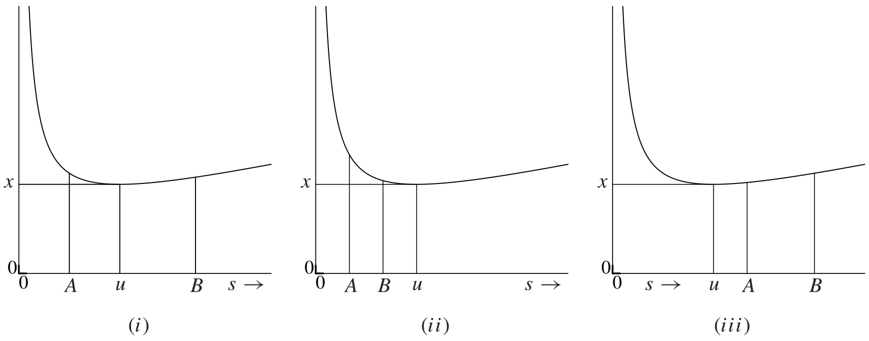
Figure captions

Fig. 1. Graph of the mapping $s \to s(\frac{1}{2}\,\alpha + \frac{1}{2}\,\beta\, s^{-1})^2$ with $\alpha > 0$, $\beta > 0$ and $0 < A < B$ such that (i) $A \leq u \leq B$, (ii) $A \leq B < u$, (iii) $u < A \leq B$. Here $u = \beta/\alpha$ and $x = \alpha\beta$ are both taken 1.

Fig. 2. Graph of the mapping $z \to z(\frac{3}{2}\,\varepsilon_0 - \frac{1}{2}\,\varepsilon_1 z^2)$ with $\varepsilon_0 > 0$, $\varepsilon_0 > 0$ and $0 < E < F$ such that (i) $E \leq v \leq F$, (ii) $E \leq F < v$, (iii) $v < E \leq F$. Here $v = (\varepsilon_0/\varepsilon_1)^{1/2}$ and $y = (\varepsilon_0^3/\varepsilon_1)^{1/2}$ are both taken 1, and $w = (3\varepsilon_0/\varepsilon_1)^{1/2}$.



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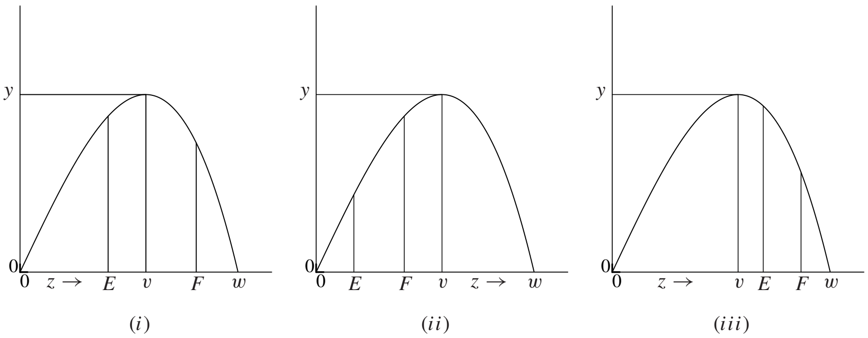

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