

Hermite function description of Feichtinger's space S_0

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Abstract.

For $\gamma \in \mathbb{R}$ we let $C(\gamma)$ be the set of all $f \in \mathcal{S}'$ for which $\sum_{n=0}^{\infty} |(f, h_n)| (n+1)^\gamma < \infty$, where $(h_n)_{n=0,1,\dots}$ is the orthonormal base of Hermite functions. We show that $C(\gamma) \subset S_0$ if and only if $\gamma \geq \frac{1}{4}$ and that $S_0 \subset C(\gamma)$ if and only if $\gamma \leq -\frac{1}{4}$. Here S_0 is Feichtinger's space.

Math Subject Classification.

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Feichtinger's space, Hermite functions, time-frequency analysis, sequence space, Bargmann transform.

1 Introduction

Feichtinger's space S_0 is the largest Banach space of functions of a real variable that allows a relatively carefree development of time-frequency analysis and, in particular, Gabor analysis, see [1], Ch. 3. There are many equivalent characterizations of the space S_0 . For our purposes, it is most convenient to work with the description in terms of the Gaussian window Short-Time Fourier Transform (STFT) of the members of S_0 . Hence, letting for $x, y \in \mathbb{R}$

$$g_{x,y}(t) = 2^{1/4} \exp(-\pi(t-x)^2 + 2\pi iyt) , \quad t \in \mathbb{R} , \quad (1)$$

we say that an $f \in L^2$ belongs to S_0 when

$$\|f\|_{S_0} := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(f, g_{x,y})| dx dy < \infty . \quad (2)$$

In phase space analysis and in the time-frequency analysis of functions in $L^2(\mathbb{R})$, the orthonormal base $(h_n)_{n=0,1,\dots}$ for $L^2(\mathbb{R})$ of Hermite functions plays an important role, see [2], Ch. 1, Sec. 7, [3], Sec. 3.4. In this paper, we aim at describing the space S_0 in terms of the Hermite coefficients (f, h_n) of its members f . To be more precise, we let for $\gamma \in \mathbb{R}$

$$C(\gamma) := \left\{ f \in \mathcal{S}' \mid \sum_{n=0}^{\infty} |(f, h_n)| (n+1)^\gamma < \infty \right\} , \quad (3)$$

where \mathcal{S}' is the set of tempered distributions (dual of Schwartz's \mathcal{S}). We shall show that $C(\frac{1}{4}) \subset S_0 \subset C(-\frac{1}{4})$ and that $C(\gamma) \not\subset S_0$ when $\gamma < \frac{1}{4}$ and $S_0 \not\subset C(\gamma)$ when $\gamma > -\frac{1}{4}$. Figure 1 summarizes the situation. It may be

noted here that from [4] one has $\mathcal{S} = \bigcap_{\gamma \in \mathbb{R}} C(\gamma)$ while $\mathcal{S}' = \bigcup_{\gamma \in \mathbb{R}} C(\gamma)$.

The main tools we use to prove these results are the Bargmann transform that allows explicit computations with the Gaussian window STFTs of the Hermite functions, and Mehler's formula. The inclusions $C(\frac{1}{4}) \subset S_0 \subset C(-\frac{1}{4})$ are rather straightforward consequences of the explicit formulas for $(h_n, g_{x,y})$ and orthonormality of the h_n , $n = 0, 1, \dots$. For the proof of $S_0 \not\subset C(\gamma)$, certain manipulations with Mehler's formula are required, while the proof of $C(\gamma) \not\subset S_0$ relies on a consideration of certain "lacunary" Hermite series $\sum_m c_m h_{A^m}$ for which one has $\|\sum_m c_m h_{A^m}\|_{S_0} \approx \sum_m |c_m| \|h_{A^m}\|_{S_0}$.

Formula (14) for the Gaussian window STFT of the Hermite functions is crucial in these investigations. This formula shows, for instance, that the n^{th} -Hermite function can be thought as being located in the time-frequency plane

near the circle around 0 of radius $((n+1)/\pi)^{1/2}$. Furthermore, as a function of $\varphi = \arg(x - iy)$ for a fixed value of $r = (x^2 + y^2)^{1/2}$, the collection of functions $(h_n, g_{x,y})$, $n = 0, 1, \dots$, is orthogonal. These circumstances lead one to expect quite transparent characterization results for modulation spaces $M_m^{p,q}$, see [3], Definition 11.1.2 on p. 219 and 11.3.1 on p. 231, with $p = q = 2$ and radially symmetric, well-behaved weight functions $m(x, y) = K(\pi(x^2 + y^2))$. Indeed, when, say $f \in \mathcal{S}$ and K has at most polynomial growth, one has the formula

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(f, g_{x,y})|^2 K(\pi(x^2 + y^2)) dx dy &= \\ &= \sum_{n=0}^{\infty} |(f, h_n)|^2 \int_0^{\infty} \frac{e^{-v} v^n}{n!} K(v) dv \end{aligned} \quad (4)$$

for the square of the $M_m^{2,2}$ -norm of f . This follows readily from the above observations on the $(h_n, g_{x,y})$ and the completeness and orthonormality of the h_n in $L^2(\mathbb{R})$. When, in addition, K is slowly varying (a notion that can be made more precise in a similar way as is done in [5], Definition 3 on p. 284), one can effectively replace the integrals on the second line of (4) by $K(n+1)$. Thus we arrive at the characterization result

$$f \in M_m^{2,2} \Leftrightarrow \sum_{n=0}^{\infty} |(f, h_n)|^2 K(n+1) < \infty, \quad (5)$$

a result that is similar in spirit to those that can be found in [5], Sec. 4.

The results for S_0 presented above show that a characterization of S_0 in terms of a single requirement on the Hermite coefficients of its members, such as is the case for $M_m^{2,2}$ -spaces with radially symmetric weights m of certain type via (5), cannot be expected to exist. This is no surprise since orthonormality is a notion that one normally associates with L^2 -norms while the norm on S_0 is an L^1 -norm, see (2). One has that S_0 coincides with the modulation space $M_{m=1}^{p=1,q=1}$. The results of this paper on S_0 thus show to what extent the crisp result in (5) gets detuned when one passes from 2-norms to 1-norms at either side of the \Leftrightarrow in (5) ($K \equiv 1$ here). The proofs of the results get more complicated as well, but the concentration result for the Gaussian STFT of the h_n 's is still crucial.

In Sec. 2 we develop notations, normalizations and relevant formulas, such as the one for $(h_n, g_{x,y})$ and Mehler's formula, concerning the Hermite functions. In Sec. 3 we show the inclusions $C(\frac{1}{4}) \subset S_0 \subset C(-\frac{1}{4})$. In Sec. 4 we show that $S_0 \not\subset C(\gamma)$ when $\gamma > -\frac{1}{4}$ and in Sec. 5 we show that $C(\gamma) \not\subset S_0$ when $\gamma < \frac{1}{4}$.

2 Preliminaries about Hermite functions

We follow the notations and normalizations of [2], p. 51; these are identical with those in [3], Sec. 3.4 and [6], except that in [3] and [6] one writes H_n and ψ_n , respectively, for h_n . Thus we let

$$h_n(t) = \frac{2^{1/4}}{\sqrt{n!}} \left(\frac{-1}{2\sqrt{\pi}} \right)^n e^{\pi t^2} \left(\frac{d}{dt} \right)^n e^{-2\pi t^2}, \quad n = 0, 1, \dots, \quad t \in \mathbb{C}, \quad (6)$$

be the Hermite functions. In terms of the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx} \right)^n e^{-x^2}, \quad n = 0, 1, \dots, \quad x \in \mathbb{C}, \quad (7)$$

one has

$$h_n(t) = (2^{n-1/2} n!)^{-1/2} H_n(t \sqrt{2\pi}) e^{-\pi t^2}, \quad n = 0, 1, \dots, \quad t \in \mathbb{C}. \quad (8)$$

See [7], Sec. 5.5 and p. 190. From [7], (8.22.8) on p. 200 we have

$$h_n(t) = \frac{2^{1/2}}{(\pi(n + \frac{1}{2}))^{1/4}} \cos[2(\pi(n + \frac{1}{2}))^{1/2} t - \frac{1}{2} \pi n] + O((n + \frac{1}{2})^{-3/4}), \quad (9)$$

where the O holds uniformly locally in $t \in \mathbb{R}$. Also, from the table on p. 700 in [8] (in which we have $h_n(t) = (2\pi)^{1/4} \mathcal{H}_n(t \sqrt{2\pi})$), we get

$$\begin{aligned} h_n(t) &= O((n+1)^{-1/4}), & -\frac{n}{\sqrt{2\pi}} \leq t \leq \frac{n}{\sqrt{2\pi}}, \\ h_n(t) &= O((n+1)^{-1/12}), & t \in \mathbb{R}. \end{aligned} \quad (10)$$

The Gaussian window STFT of the Hermite functions can be computed by using properties of the Bargmann transform, see [2], p. 40 and [3], Sec. 3.4. With $f \in L^2(\mathbb{R})$, the Bargmann transform Bf of f is defined as

$$(Bf)(z) = e^{-\frac{1}{2}\pi z^2} \int_{-\infty}^{\infty} f(t) 2^{1/4} e^{-\pi(t-z)^2} dt, \quad z \in \mathbb{C}. \quad (11)$$

There holds for $f \in L^2(\mathbb{R})$

$$(f, g_{x,y}) = e^{-\frac{1}{2}\pi(x^2+y^2)-\pi ixy} (Bf)(x-iy), \quad x, y \in \mathbb{R}. \quad (12)$$

From the generating formula, [2], (1.86) on p. 54, for the Hermite functions and orthonormality it follows that

$$(Bh_n)(z) = \frac{(\pi^{1/2} z)^n}{(n!)^{1/2}}, \quad n = 0, 1, \dots, \quad z \in \mathbb{C}. \quad (13)$$

Consequently, one has

$$(h_n, g_{x,y}) = e^{-\frac{1}{2}\pi(x^2+y^2)-\pi ixy} \frac{(\pi^{1/2}(x-iy))^n}{(n!)^{1/2}}, \quad n = 0, 1, \dots, \quad x, y \in \mathbb{R}. \quad (14)$$

We then compute, for instance,

$$\begin{aligned} \|h_n\|_{S_0} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\pi(x^2+y^2)} \frac{(\pi^{1/2}(x^2+y^2)^{1/2})^n}{(n!)^{1/2}} dx dy = \\ &= 2^{\frac{1}{2}n+1} \frac{\Gamma(\frac{1}{2}n+1)}{(n!)^{1/2}}, \quad n = 0, 1, \dots, \end{aligned} \quad (15)$$

in which we have used polar coordinates $x+iy = r e^{i\varphi}$ to evaluate the double integral.

Mehler's formula can be found in [7], Problem 23 on p. 380. In terms of the Hermite functions h_n it takes the form as can be found in [2], (1.87) on p. 55: for $t, s \in \mathbb{C}$, $w \in \mathbb{C}$, $|w| < 1$ there holds

$$\sum_{n=0}^{\infty} w^n h_n(t) h_n(s) = \left(\frac{2}{1-w^2} \right)^{1/2} \exp\left(\frac{4\pi t s w - \pi(t^2 + s^2)(1+w^2)}{1-w^2} \right). \quad (16)$$

3 Proof that $C(\frac{1}{4}) \subset S_0 \subset C(-\frac{1}{4})$

We shall first show that $C(\frac{1}{4}) \subset S_0$. It follows from Stirling's formula $\Gamma(x+1) = x^{x+1/2} e^{-x} \sqrt{2\pi} (1 + O(\frac{1}{x}))$, see [9], 6.1.37–38 on p. 257, and (15) that

$$\|h_n\|_{S_0} = 2(\frac{1}{2}\pi n)^{1/4} (1 + O(n^{-1})), \quad n = 0, 1, \dots. \quad (17)$$

Since S_0 is a Banach space, we have for $f \in \mathcal{S}'$ that

$$\sum_{n=0}^{\infty} |(f, h_n)| (n+1)^{1/4} < \infty \Rightarrow f = \sum_{n=0}^{\infty} (f, h_n) h_n \in S_0. \quad (18)$$

Hence, when $f \in C(\frac{1}{4})$ we have that $f \in S_0$, and the Hermite series of f converges in the sense of S_0 to f .

We next show that $S_0 \subset C_0(-\frac{1}{4})$. We let $f \in L^2(\mathbb{R})$, and we apply the inversion formula for the STFT, see [3], Corollary 3.2.3 on p. 44, to obtain

$$|(f, h_n)| = \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, g_{x,y})(g_{x,y}, h_n) dx dy \right| \leq$$

$$\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(f, g_{x,y})| e^{-\frac{1}{2}\pi(x^2+y^2)} \frac{(\pi^{1/2}(x^2+y^2)^{1/2})^n}{(n!)^{1/2}} dx dy, \quad (19)$$

where we have used (14). Assume that $f \in S_0$. We must show that $\sum_n |(f, h_n)| (n+1)^{-1/4} < \infty$. To that end we shall prove below that there is a $C > 0$ such that

$$F(v) := \sum_{n=0}^{\infty} \frac{v^{n/2}}{(n!)^{1/2}(n+1)^{1/4}} \leq C e^{\frac{1}{2}v}, \quad v \geq 0. \quad (20)$$

It follows from (20) that

$$\sum_{n=0}^{\infty} |(f, h_n)| (n+1)^{-1/4} \leq C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(f, g_{x,y})| dx dy = C \|f\|_{S_0}, \quad (21)$$

which implies that $f \in C(-\frac{1}{4})$.

We shall now show that (20) holds for some $C > 0$. To that end we prove that there are $c_1 > 0$, $c_2 > 0$ such that

$$\frac{v^x e^{-v}}{\Gamma(x+1)} \leq \frac{1}{\sqrt{2\pi x}} \cdot \begin{cases} \exp\left(-\frac{c_1}{x}(v-x)^2\right), & \frac{1}{2}x \leq v \leq 2x, \\ \exp(-c_2|v-x|), & 0 \leq x \leq v \text{ or } v \geq 2x. \end{cases} \quad (22)$$

To show (22), we first note that from the inequality $\Gamma(x+1) \geq x^x e^{-x} \sqrt{2\pi x}$, $x \geq 0$, see [10], bottom half of p. 251, we have that

$$\frac{v^x e^{-v}}{\Gamma(x+1)} \leq \frac{1}{\sqrt{2\pi x}} \exp\left[x\left(\ln \frac{v}{x} + 1 - \frac{v}{x}\right)\right], \quad x, v \geq 0. \quad (23)$$

The function $w > 0 \rightarrow r(w) := \ln w + 1 - w$ is smooth and concave, and has its maximum 0 at $w = 1$. Consequently, there are $c_1 > 0$, $c_2 > 0$ such that

$$r(w) \leq -c_1(w-1)^2, \quad \frac{1}{2} \leq w \leq 2, \quad (24)$$

$$r(w) \leq -c_2|w-1|, \quad 0 < w \leq \frac{1}{2} \text{ or } w \geq 2. \quad (25)$$

From this (22) follows. Therefore, when $v \geq 1$, we have

$$e^{-\frac{1}{2}v} F(v) = \sum_{n=0}^{\infty} \left(\frac{v^n e^{-v}}{n!}\right)^{1/2} \frac{1}{(n+1)^{1/4}} \leq$$

$$\begin{aligned}
&\leq e^{-\frac{1}{2}v} + \left(\sum_{1 \leq n \leq \frac{1}{2}v} + \sum_{n \geq 2v} \right) \left(\frac{1}{2\pi n(n+1)} \right)^{1/4} \exp(-\frac{1}{2}c_2|v-n|) + \\
&\quad + \sum_{\frac{1}{2}v \leq n \leq 2v} \left(\frac{1}{2\pi n(n+1)} \right)^{1/4} \exp\left(-\frac{c_1}{2n}(v-n)^2\right). \tag{26}
\end{aligned}$$

The first term on the second line of (26), $\exp(-\frac{1}{2}v)$, is bounded in $v \geq 0$. The second term is bounded by $(4/\pi)^{1/4} (1 - \exp(-\frac{1}{2}c_2))^{-1} \exp(-\frac{1}{4}c_2v)$, and the term on the last line is bounded by

$$\begin{aligned}
&\left(\frac{2}{\pi v(v+2)} \right)^{1/4} \sum_{n=-\infty}^{\infty} \exp(-\frac{1}{4}c_1v^{-1}(v-n)^2) = \\
&= \left(\frac{2}{\pi v(v+2)} \right)^{1/4} \left(\frac{4\pi v}{c_1} \right)^{1/2} \sum_{m=-\infty}^{\infty} \exp[-4\pi^2 m^2 v/c_1 + 2\pi i m v] \leq \\
&\leq \left(\frac{32\pi}{c_1^2} \frac{v}{v+2} \right)^{1/4} \sum_{m=-\infty}^{\infty} \exp(-4\pi^2 m^2 v/c_1) \leq \\
&\leq \left(\frac{32\pi}{c_1^2} \right)^{1/4} \sum_{m=-\infty}^{\infty} \exp(-4\pi^2 m^2/c_1) \tag{27}
\end{aligned}$$

when $v \geq 1$. Here we have used the Poisson summation formula for the first identity in (27). This shows (20).

Notes.

1. In a similar fashion one can prove the existence of a $D > 0$ such that

$$\sum_{n=0}^{\infty} |(f, h_n)| \leq D \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(f, g_{x,y})| (1+x^2+y^2)^{1/4} dx dy \tag{28}$$

holds for any $f \in L^2(\mathbb{R})$. Furthermore, many variations on this theme are possible.

2. From what we show in Sec. 5 it follows that $C(-\frac{1}{4}) \not\subset S_0$. Nevertheless, the members of $C(-\frac{1}{4})$ are reasonably well-behaved functions. Indeed, it

follows from (9) and/or (10) that the function f and its Fourier transform $\mathcal{F}f$,

$$f = \sum_{n=0}^{\infty} (f, h_n) h_n, \quad \mathcal{F}f = \sum_{n=0}^{\infty} i^n (f, h_n) h_n \quad (29)$$

are continuous, the series being locally uniformly convergent when $f \in C(-\frac{1}{4})$.

4 Proof that $S_0 \not\subset C(\gamma)$, $\gamma > -\frac{1}{4}$

We shall now show that $S_0 \not\subset C(\gamma)$ when $-\frac{1}{4} < \gamma < \frac{1}{4}$. For this we use Mehler's formula (16) with $s = 0$ and $w = iv$, $0 \leq v < 1$, so that for $0 \leq v < 1$, $t \in \mathbb{R}$

$$\sum_{n=0}^{\infty} i^n v^n h_n(0) h_n(t) = \left(\frac{2}{1+v^2} \right)^{1/2} \exp\left(-\pi \frac{1-v^2}{1+v^2} t^2 \right). \quad (30)$$

We observe from [7], 5.5.4–5 on p. 106 that $h_{2m+1}(0) = 0$, $m = 0, 1, \dots$, while

$$(-1)^m h_{2m}(0) = \left(2^{-2m+1/2} \frac{(2m)!}{m! m!} \right)^{1/2}, \quad m = 0, 1, \dots \quad (31)$$

From Appendix A it follows that

$$\left(\frac{2}{\pi(m + \frac{1}{2})} \right)^{1/4} \leq (-1)^m h_{2m}(0) \leq \left(\frac{2}{\pi(m + \frac{1}{4})} \right)^{1/4}, \quad m = 0, 1, \dots \quad (32)$$

Now let $\alpha \in (\frac{1}{2}, 1]$, multiply (30) by $(1-v)^{\alpha-1}$, and integrate over $v \in [0, 1)$. There results

$$\begin{aligned} & \sum_{m=0}^{\infty} (-1)^m h_{2m}(0) B(\alpha, 2m+1) h_{2m}(t) = \\ & = \int_0^1 \left(\frac{2}{1+v^2} \right)^{1/2} \exp\left(-\pi \frac{1-v^2}{1+v^2} t^2 \right) (1-v)^{\alpha-1} dv =: f_{\alpha}(t), \quad t \in \mathbb{R}, \end{aligned} \quad (33)$$

where for $m = 0, 1, \dots$

$$B(\alpha, 2m+1) = \int_0^1 (1-v)^{\alpha-1} v^{2m} dv = \frac{\Gamma(\alpha) \Gamma(2m+1)}{\Gamma(2m+1+\alpha)}. \quad (34)$$

In (33) we have interchanged the summation over $m = 0, 1, \dots$ and the integration over $v \in [0, 1)$. This is justifiable by (9) and/or (10), (32), and the fact that the integrand in (34) is non-negative while by Appendix A

$$\Gamma(\alpha)(2m+1)^{-\alpha} \leq B(\alpha, 2m+1) \leq \Gamma(\alpha)(2m + \frac{1}{2}\alpha + \frac{1}{2})^{-\alpha}. \quad (35)$$

We also see from (32) and (35) that $f_\alpha \in C(\gamma)$ if and only if $\alpha > \frac{3}{4} + \gamma$.

We shall now show that f_α in (33), with $\alpha > \frac{1}{2}$, is in S_0 . To that end we compute for $\beta > 0$ the Gaussian window STFT of $\exp(-\pi\beta t^2)$ as

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{-\pi\beta t^2} 2^{1/4} e^{-\pi(t-x)^2 - 2\pi iyt} dt = \\ & = 2^{1/4} \frac{1}{\sqrt{1+\beta}} \exp\left[\frac{-\pi\beta}{1+\beta}x^2 - \frac{\pi}{1+\beta}y^2 - \frac{2\pi i}{1+\beta}xy\right], \quad x, y \in \mathbb{R}, \end{aligned} \quad (36)$$

and the S_0 -norm of $\exp(-\pi\beta t^2)$ as

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{2^{1/4}}{\sqrt{1+\beta}} \exp\left[\frac{-\pi\beta}{1+\beta}x^2 - \frac{\pi}{1+\beta}y^2\right] dx dy = 2^{1/4} \left(\frac{1+\beta}{\beta}\right)^{1/2}. \quad (37)$$

It follows that

$$\|f_\alpha\|_{S_0} \leq \int_0^1 \left(\frac{2}{1+v^2}\right)^{1/2} (1-v)^{\alpha-1} 2^{1/4} \left(\frac{2}{1-v^2}\right)^{1/2} dv < \infty \quad (38)$$

since $\alpha > \frac{1}{2}$. Conversely, when we consider f_α in (33) with $\alpha > 0$, we have that

$$\int_{-\infty}^{\infty} f_\alpha(t) dt = \int_0^1 \left(\frac{2}{1+v}\right)^{1/2} (1-v)^{\alpha-3/2} dv < \infty \quad (39)$$

if and only if $\alpha > \frac{1}{2}$. Thus $f_\alpha \in S_0$ if and only if $\alpha > \frac{1}{2}$.

We conclude that for $\gamma > -\frac{1}{4}$ we have $f_\alpha \in S_0 \setminus C(\gamma)$ if and only if $\frac{1}{2} < \alpha \leq \frac{3}{4} + \gamma$. In particular, $S_0 \setminus C(\gamma)$ is non-empty when $\gamma > -\frac{1}{4}$. At this point it is interesting to ask whether perhaps the Hermite series for f_α , see (33), converges in S_0 -sense when $\frac{1}{2} < \alpha \leq 1$. This is indeed so, but the proof is quite technical and so we omit it.

5 Proof that $C(\gamma) \not\subset S_0$, $\gamma < \frac{1}{4}$

We start by noting that $C(\gamma_1) \subset C(\gamma_2)$ when $\gamma_1 > \gamma_2$, whence it is sufficient to consider $0 < \gamma < \frac{1}{4}$. We shall consider $f \in L^2(\mathbb{R})$ of the form

$$f = \sum_{m=0}^{\infty} c_m h_{K_m} , \quad (40)$$

where we take $K_m = A^m$ with integer $A \geq 4$ and $c_m = (K_m + 1)^{-1/4}$. This f satisfies

$$\sum_{n=0}^{\infty} |(f, h_n)| (n+1)^\gamma = \sum_{m=0}^{\infty} (A^m + 1)^{\gamma-1/4} < \infty . \quad (41)$$

Hence $f \in C(\gamma)$ for $\gamma < \frac{1}{4}$, and certainly $f \in L^2(\mathbb{R})$.

We compute, using (14), for this f

$$\begin{aligned} \|f\|_{S_0} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\pi(x^2+y^2)} \left| \sum_{m=0}^{\infty} c_m \frac{(\pi^{1/2}(x-iy))^{K_m}}{(K_m!)^{1/2}} \right| dx dy = \\ &= \frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} \left| \sum_{m=0}^{\infty} c_m f_m(v) e^{-iK_m\varphi} \right| dv d\varphi , \end{aligned} \quad (42)$$

where we have used polar coordinates $x + iy = r e^{i\varphi}$ and subsequently substituted $v = \pi r^2$, and where f_m is defined by

$$f_m(v) = \left(\frac{v^{K_m} e^{-v}}{K_m!} \right)^{1/2} , \quad m = 0, 1, \dots , \quad v \geq 0 . \quad (43)$$

Since $A \geq 4$, $K_m = A^m$, we have that the intervals $(\frac{1}{2} K_m, 2K_m)$ are non-overlapping. Consequently,

$$\begin{aligned} \|f\|_{S_0} &\geq \frac{1}{2\pi} \sum_{m=0}^{\infty} \int_{\frac{1}{2}K_m}^{2K_m} \int_0^{2\pi} \left(c_m f_m(v) - \sum_{n \neq m} c_n f_n(v) \right) dv d\varphi = \\ &= \sum_{m=0}^{\infty} I_{mm} - \sum_{m=0}^{\infty} \sum_{n=0, n \neq m}^{\infty} I_{mn} = \\ &= \sum_{m=0}^{\infty} I_{mm} - \sum_{n=0}^{\infty} \sum_{m=0, m \neq n}^{\infty} I_{mn} , \end{aligned} \quad (44)$$

where

$$I_{mn} = \int_{\frac{1}{2}K_m}^{2K_m} c_n f_n(v) dv, \quad m, n = 0, 1, \dots \quad (45)$$

For $m = 0, 1, \dots$, there holds

$$I_{mm} = \int_0^{\infty} c_m f_m(v) dv - \int_{v \notin (\frac{1}{2}K_m, 2K_m)} c_m f_m(v) dv, \quad (46)$$

while for $n = 0, 1, \dots$, there holds

$$\sum_{m=0, m \neq n}^{\infty} I_{mn} = \sum_{m=0, m \neq n}^{\infty} \int_{\frac{1}{2}K_m}^{2K_m} c_n f_n(v) dv \leq \int_{v \notin (\frac{1}{2}K_n, 2K_n)} c_n f_n(v) dv. \quad (47)$$

Hence

$$\|f\|_{s_0} \geq \sum_{m=0}^{\infty} \int_0^{\infty} c_m f_m(v) dv - 2 \sum_{m=0}^{\infty} \int_{v \notin (\frac{1}{2}K_m, 2K_m)} c_m f_m(v) dv. \quad (48)$$

Now from (43) and (22) we see that

$$\int_{v \notin (\frac{1}{2}K_m, 2K_m)} f_m(v) dv \leq \frac{1}{\sqrt{2\pi K_m}} \int_{-\infty}^{\infty} \exp(-\frac{1}{2} c_2 |v - K_m|) dv = \frac{4}{c_2 \sqrt{2\pi K_m}}. \quad (49)$$

With $c_m = (K_m + 1)^{-1/4}$, $K_m = A^m$, $A \geq 4$, we then see that the second series at the right-hand side of (48) is finite. On the other hand, we have from (43)

$$\int_0^{\infty} f_m(v) dv = \left(\frac{1}{K_m!}\right)^{1/2} \int_0^{\infty} v^{\frac{1}{2}K_m} e^{-\frac{1}{2}v} dv = 2 \left(\frac{2^{K_m} \Gamma^2(\frac{1}{2}K_m + 1)}{\Gamma(K_m + 1)}\right)^{1/2}. \quad (50)$$

By Appendix A there holds for $y > 0$

$$\frac{2^y \Gamma^2(\frac{1}{2}y + 1)}{\Gamma(y + 1)} \geq \sqrt{\frac{1}{2}\pi y}. \quad (51)$$

Therefore

$$\int_0^{\infty} f_m(v) dv \geq 2^{3/4} \pi^{1/4} K_m^{1/4}, \quad (52)$$

whence the first series at the right-hand side of (48) equals ∞ since $c_m = (K_m + 1)^{-1/4}$. Therefore, $\|f\|_{S_0} = \infty$ and thus $f \notin S_0$. Since $f \in C(\gamma)$ when $\gamma < \frac{1}{4}$, we see that $C(\gamma) \not\subset S_0$, and the proof is complete.

A Appendix. Some lemmas about B -functions

In the main text we needed for $m = 0, 1, \dots$, $\alpha \in (0, 1)$, $y > 0$

$$\frac{1}{\sqrt{\pi(m + \frac{1}{2})}} \leq 2^{-2m} \frac{(2m)!}{m! m!} \leq \frac{1}{\sqrt{\pi(m + \frac{1}{4})}}, \quad (53)$$

$$\Gamma(\alpha)(2m + 1)^{-\alpha} \leq B(\alpha, 2m + 1) \leq \Gamma(\alpha)(2m + \frac{1}{2} \alpha + \frac{1}{2})^{-\alpha}, \quad (54)$$

$$\frac{2^y \Gamma^2(\frac{1}{2} y + 1)}{\Gamma(y + 1)} = \frac{1}{2} (y + 1) B(\frac{1}{2}, \frac{1}{2} y + 1) \geq \sqrt{\frac{1}{2} \pi y}, \quad (55)$$

where the B -function is given as

$$B(\alpha, x + 1) = \int_0^1 (1 - v)^{\alpha-1} v^x dv = \frac{\Gamma(\alpha) \Gamma(x + 1)}{\Gamma(x + 1 + \alpha)}. \quad (56)$$

We first observe that

$$2^{-2m} \frac{(2m)!}{m! m!} = \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m + 1) \Gamma(\frac{1}{2})} = \frac{1}{\pi} B(\frac{1}{2}, m + \frac{1}{2}), \quad (57)$$

and that for $x > -1$

$$\frac{2^{2x} \Gamma^2(x + 1)}{\Gamma(2x + 1)} = 2^{2x} (2x + 1) B(x + 1, x + 1), \quad (58)$$

while

$$\begin{aligned} 2^{2x} B(x + 1, x + 1) &= 2^{2x} \int_0^1 (t(1 - t))^x dt = \int_0^1 (1 - (1 - 2t)^2)^x dt = \\ &= \frac{1}{2} \int_0^1 (1 - s)^x s^{-1/2} ds = \frac{1}{2} B(\frac{1}{2}, x + 1). \end{aligned} \quad (59)$$

It thus suffices to show that for $\alpha \in (0, 1]$, $x > -1$

$$\Gamma(\alpha)(x+1)^{-\alpha} \leq B(\alpha, x+1) \leq \Gamma(\alpha)(x + \frac{1}{2}\alpha + \frac{1}{2})^{-\alpha}. \quad (60)$$

We have by the substitution $v = e^{-t}$, $t \in [0, \infty)$

$$\begin{aligned} B(\alpha, x+1) &= \int_0^1 (1-v)^{\alpha-1} v^x dv = \int_0^\infty (1-e^{-t})^{\alpha-1} e^{-(x+1)t} dt = \\ &= \int_0^\infty t^{\alpha-1} \left(\frac{\sinh(t/2)}{t/2} \right)^{\alpha-1} e^{-(x+\frac{1}{2}\alpha+\frac{1}{2})t} dt. \end{aligned} \quad (61)$$

Since for $t \geq 0$, $b > 0$

$$1 - e^{-t} \leq t, \quad \frac{\sinh(t/2)}{t/2} \geq 1, \quad \int_0^\infty t^{\alpha-1} e^{-bt} dt = b^{-\alpha} \Gamma(\alpha), \quad (62)$$

we get (60).

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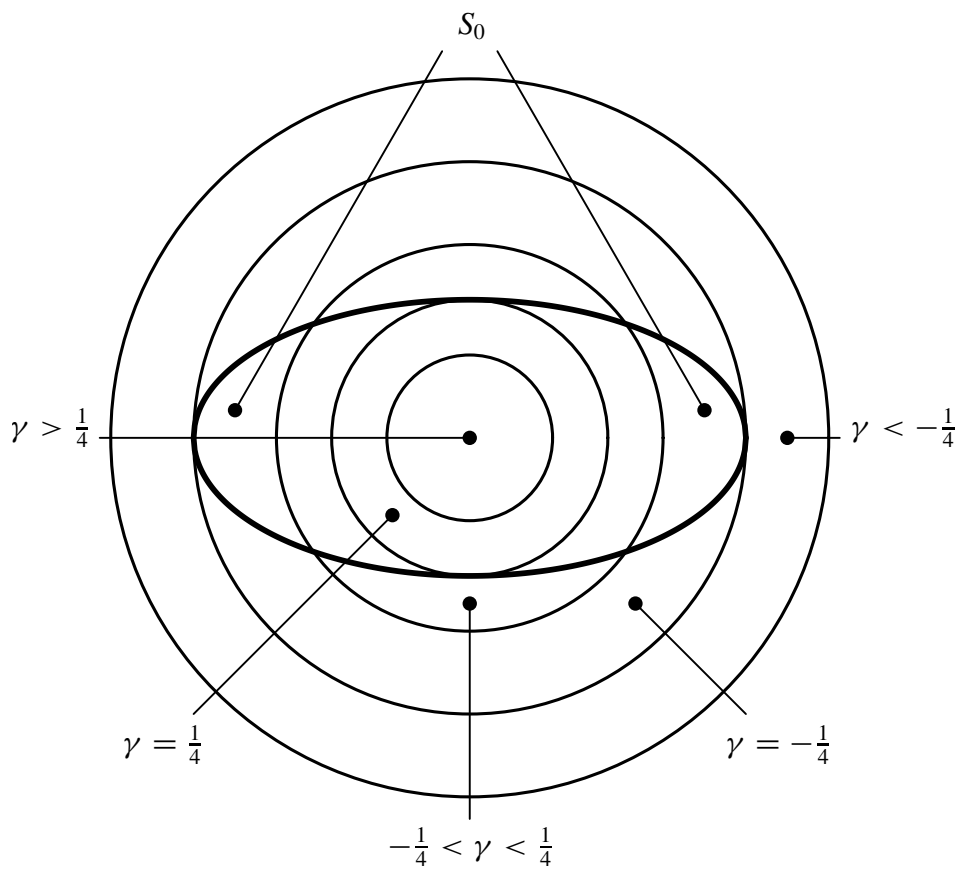


Figure 1: Positioning of S_0 relative to $C(\gamma)$, $\gamma \in \mathbb{R}$.