Zak transform characterization of S_0

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Abstract

We present a characterization of the modulation space S_0 in terms of the Zak transform of its elements. We illustrate our result by considering $S^{-\lambda}h$, where h is the standard Gaussian, S is the "frame" operator corresponding to the critical-density Gabor system $(h, a = 1, b = 1)$, and $\lambda \in [0, \frac{3}{2})$. Both the proof of the main result and the example require basics from Gabor frame theory; these are developed in a separate section. We further use a result from recent work by Gröchenig and Leinert on Wiener-type theorems in a non-commutative setting. We also present an extension of our main result to more general modulation spaces.

Key words and phrases : Feichtinger space S_0, Zak transform, sampled short-time Fourier transform, Gabor system, critical density, modulation space.

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1 Introduction

It is now 25 years ago that Hans Feichtinger introduced the modulation space S_0, see [9]. This space turned out to be a very convenient and sufficiently large space of functions for doing time-frequency analysis, and in particular Gabor analysis, without being bothered by measure-theoretic intricacies that hamper the developments in an L^2-setting. We refer to [12] for an extensive survey of the many properties and characterizations of S_0 that are desirable from a time-frequency analyst's point of view. Around the same time that S_0 was invented, the Zak transform entered the field of time-frequency analysis as a tool for studying Gabor expansions at critical density, see [4], [16], [17] and [18]. For historical surveys with attention to the several places where the Zak transform (Weil-Brezin mapping, kq representation) occurs in the literature, we refer to [19], [13], [14]. Already in [3], [2], the Zak transform appears implicitly or explicitly as a tool for proving completeness and expansion results for the case of Gaussian windows at critical density (Von Neumann lattices). The introduction of S_0 and the occurrence of the Zak transform were events that contributed considerably to the genesis of the field of mathematical time-frequency analysis; the textbook [14] is entirely devoted to this field.

In this paper, we take the definition of S_0 that is based on the short-time Fourier transform (STFT) as our starting point. When $f, g \in L^2(\mathbb{R})$, the short-time Fourier transform of f using the window g is defined by

$$(S_g f)(x,y) = \int_{-\infty}^{\infty} e^{-2\pi i y t}\, f(t)\, g^*(t-x)\, dt = (f, g_{x,y}) \ , \qquad x, y \in \mathbb{R} \ , \tag{1.1}$$

where * denotes complex conjugation and

$$g_{x,y}(t) = e^{2\pi i y t}\, g(t-x) \ , \quad t \in \mathbb{R} \ ; \qquad x, y \in \mathbb{R} \ . \tag{1.2}$$

Now let $g \in \mathcal{S}$ (Schwartz space), $g \neq 0$. Then an $f \in L^2(\mathbb{R})$ belongs to S_0 if and only if $S_g f \in L^1(\mathbb{R}^2)$. For deciding whether an $f \in L^2(\mathbb{R})$ belongs to S_0, one can use any $g \in S_0$ with $g \neq 0$: when $f \in L^2(\mathbb{R})$, $g \in S_0$, $g \neq 0$ we have $f \in S_0$ if and only if $S_g f \in L^1(\mathbb{R}^2)$.

The Zak transform of an $f \in \mathcal{S}$ is defined by

$$(Zf)(t,\nu) = \sum_{k=-\infty}^{\infty} f(t-k)\, e^{2\pi i k \nu} \ , \qquad t, \nu \in \mathbb{R} \ . \tag{1.3}$$

In Subsec. 2.2 we present the properties of the Zak transform as far as they are relevant to our present purposes. We have for $f, g \in \mathcal{S}$ that $Zf(Zg)^*$ is 1-periodic in its two variables, and there holds for $n, m \in \mathbb{Z}$,

$$\int_0^1 \int_0^1 (Zf)(t,\nu)(Zg)^*(t,\nu)\, e^{-2\pi i n t - 2\pi i m \nu}\, dt\, d\nu = (f, g_{-m,n}) \ . \tag{1.4}$$

The formula (1.4) represents a direct link between Zak transform theory and (sampled) short-time Fourier transforms as used in the definition of S_0. In fact, this connection has been used in [26] to obtain characterizations of modulation spaces and lattice size estimates ensuring Gabor systems to be Gabor frames. The question we concern ourselves with in this paper is whether one can tell for an $f \in L^2(\mathbb{R})$ its membership of S_0 by inspecting functions $Zf(Zg)^*$ with $g \in S_0$. Here one could hope that inspection of one such function is enough. However, when $g \in S_0$ we have that Zg is continuous and vanishes at least at one point $(t_0, \nu_0) \in [0,1)^2$. Consequently, $Zf(Zg)^*$ could be reasonably behaved at (t_0, ν_0) even though Zf itself is not. The main result that we will show is as follows. Let $g^{(1)}, g^{(2)} \in S_0$ be such that their Zak transforms have no common zeros, and let $f \in L^2(\mathbb{R})$. Then $f \in S_0$ if and only if $Zf(Zg^{(1)})^*$ and $Zf(Zg^{(2)})^*$

have absolutely convergent Fourier series. The proof of this result uses some facts from S_0-theory, such as a theorem on the sampling of short-time Fourier transforms $S_g f$ when both $f, g \in S_0$, some basic Gabor frame theory, and results of the Wiener-type obtained recently by Gröchenig and Leinert in the context of Gabor frames.

We illustrate our result as follows. We let

$$h(t) = 2^{1/4} \exp(-\pi t^2) \ , \qquad t \in \mathbb{R} \ , \tag{1.5}$$

be the standard Gaussian, and we consider the Gabor system (critical density)

$$(h, a = 1, b = 1) = (e^{2\pi i m t}\, h(t-n))_{n,m\in\mathbb{Z}} \ . \tag{1.6}$$

In Sec. 3 we present basic facts about Gabor systems. Although the frame operator S corresponding to the Gabor system in (1.6) is not invertible, we can consider $f^{\lambda} = S^{-\lambda} h$, with $\lambda \geq 0$, via the Zak transform domain. According to the functional calculus for frame operators in the Zak transform domain, see [14], Sec. 8.3, we have, at least formally,

$$Z(S^{-\lambda} h) = Zh / |Zh|^{2\lambda} \ . \tag{1.7}$$

The function Zh is very well-behaved, has exactly one zero in $[0,1)^2$, viz. at $(t,\nu) = (\frac{1}{2}, \frac{1}{2})$, while $\partial Zh/\partial t$, $\partial Zh/\partial \nu$ do not vanish at $(t,\nu) = (\frac{1}{2}, \frac{1}{2})$. Consequently, the right-hand side of (1.7) is in $L^1([0,1)^2)$ when $\lambda \in [0, \frac{3}{2})$ and in $L^2([0,1)^2)$ when $\lambda \in [0,1)$. This right-hand side of (1.7) can be considered as the Zak transform of f^{λ}, and upon applying the inverse Zak transform formally, we are led to the definition

$$f^{\lambda}(t) := Z^{-1}(Zh/|Zh|^{2\lambda})(t) = \int_0^1 \frac{(Zh)(t,\nu)}{|(Zh)(t,\nu)|^{2\lambda}}\, d\nu \ , \qquad t \in \mathbb{R} \ . \tag{1.8}$$

Using, among other things, our main result, we show that

$$0 \leq \lambda < \tfrac{1}{2} \quad \Rightarrow \quad f^{\lambda} \in S_0 \ , \tag{1.9}$$

$$\tfrac{1}{2} \leq \lambda < 1 \quad \Rightarrow \quad f^{\lambda} \notin S_0 \, , \quad f^{\lambda} \in L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R}) \ , \tag{1.10}$$

$$1 < \lambda < \tfrac{3}{2} \quad \Rightarrow \quad f^{\lambda} \notin L^1(\mathbb{R}) \, , \quad f^{\lambda} \notin L^{\infty}(\mathbb{R}) \ . \tag{1.11}$$

According to [18], Sec. 4.4, we have $f^{\lambda=1} \in L^{\infty}(\mathbb{R}) \backslash L^p(\mathbb{R})$ when $1 \leq p < \infty$.

The f^{λ} per se are not particularly useful for applications since they are not very well-behaved in terms of smoothness and decay. The case $\lambda = 1$ yields what has become known as Bastiaans' singular function (see [4] and [16], [17]), i.e. the dual window corresponding to the Gabor system (h, $a = 1$, $b = 1$) in a formal sense. The case $\lambda = \frac{1}{2}$ yields the tight window associated to (h, $a=1$, $b=1$) via the Zak transform as in [21]. We have $f^{1/2} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $f^{1/2}$ is just outside S_0. Furthermore, $f^{1/2}$ is Fourier invariant and ($f^{1/2}$, $a=1$, $b=1$) is an orthonormal base for $L^2(\mathbb{R})$. Hence $f^{1/2}$ competes with functions like $\chi_{(-1/2,1/2)}$ or $\mathrm{sinc}\,\pi t$ in this latter respect. In Fig. 1 we have shown a plot of f^1 and $f^{1/2}$ in the range $|t| \le 6$, and in Fig. 2 we show a plot of $f^{1/4}$ and $f^{3/4}$ in the range $|t| \le 6$.

The remainder of this paper is organized as follows. In Sec. 2 we collect what we need here from S_0-theory and the Zak transform, and in Sec. 3 we give basic facts from Gabor frame theory. There are nowadays excellent textbooks amply covering the material presented in Secs. 2, 3, see [10], [11], [14], [6], [5], so we shall be brief on this point. In Sec. 4 we present the proof of our main result. In Sec. 5 we illustrate the main result by considering f^λ defined in (1.8). Finally, in Sec. 6 we give an extension, kindly observed to us by K. Gröchenig, of our main result to more general modulation spaces.

2 Preliminaries

2.1 Preliminaries about S_0

We present results about S_0 as far as they are relevant to the purposes of the present paper. We recall that an $f \in L^2(\mathbb{R})$ belongs to S_0 if and only if

$$S_g f \in L^1(\mathbb{R}^2)\,; \quad (S_g f)(x,y) = (f, g_{x,y})\ , \qquad x,y \in \mathbb{R}\ , \tag{2.1}$$

where g is any member of S_0 with $g \neq 0$. The space S_0 is a Banach space when we take the L^1-norm of $S_g f$ as the norm of $f \in S_0$; for definiteness, we take here $g = h$, with h the standard Gaussian in (1.5). Hence, for $f \in S_0$,

$$\|f\|_{S_0} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |(f, h_{x,y})|\,dx\,dy\ , \quad h(t) = 2^{1/4}\,e^{-\pi t^2}\ , \quad t \in \mathbb{R}\ . \tag{2.2}$$

There is the following result on sampling of a short-time Fourier transform $S_g f$ when $f, g \in S_0$. Let $a > 0$, $b > 0$. We have $\|S_g g\|_{L^1} < \infty$, and there is a $C(a,b)$ such that for all $f \in S_0$ there holds

$$\sum_{n,m} |(f, g_{na,mb})| \le C(a,b)\,\|S_g g\|_{L^1}\,\|f\|_{S_0}\ . \tag{2.3}$$

See [14], Sec. 12.2 for details.

2.2 Preliminaries about Zak transforms

We next list the results needed from Zak transform theory. Below we first let $f, g \in \mathcal{S}$ (Schwartz class) and $x, y \in \mathbb{R}$, $t, \nu \in \mathbb{R}$, $n, m \in \mathbb{Z}$; we define the Zak transform of f as

$$(Zf)(t,\nu) = \sum_{k=-\infty}^{\infty} f(t-k)\, e^{2\pi i k\nu} \ . \qquad (2.4)$$

There holds

A. $\displaystyle\int_0^1\int_0^1 (Zf)(t,\nu)(Zg)^*(t,\nu)\, dt\, d\nu = \int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt$,

B. $(Zf)(t+1,\nu) = e^{2\pi i\nu}(Zf)(t,\nu)\,, \quad (Zf)(t,\nu+1) = (Zf)(t,\nu)$,

C. $(Zf_{x,y})(t,\nu) = e^{2\pi i y t}(Zf)(t-x,\nu-y)$,

D. $\displaystyle\int_0^1 (Zf)(t,\nu)\, d\nu = f(t)$,

E. $\displaystyle\int_0^1\int_0^1 (Zf)(t,\nu)(Zg)^*(t,\nu)\, e^{-2\pi i n t - 2\pi i m\nu}\, dt\, d\nu = (f, g_{-m,n})$,

F. $(Z\mathcal{F}f)(t,\nu) = e^{2\pi i\nu t}(Zf)(-\nu,t)$.

In F we denote by $\mathcal{F}f$ the Fourier transform $\int_{-\infty}^{\infty} e^{-2\pi i\nu t} f(t)\, dt$ of f.

Property A shows that $f \to Zf$ extends to an L^2-norm preserving mapping from $L^2(\mathbb{R})$ into $L^2([0,1)^2)$. Property B shows that Zf is determined by its values on any unit square. Property C reflects the basic time-frequency shift operations in Zak transform terms. Property D shows how to recover an f from its Zak transform. Property E shows how the Fourier coefficients of the $(1,1)$-periodic function $Zf(Zg)^*$ arise. Property F shows how the action of the Fourier transform is represented in Zak transform terms.

There is also the following property.

G. Assume that $Z_0 \in C^\infty(\mathbb{R}^2)$ satisfies the quasi-periodicity relations

$$Z_0(t+1,\nu) = e^{2\pi i\nu}\, Z_0(t,\nu)\,, \quad Z_0(t+1,\nu) = Z_0(t,\nu)\ , \qquad t,\nu \in \mathbb{R}\ . \quad (2.5)$$

Then there is a unique $f \in \mathcal{S}(\mathbb{R})$ such that $Z_0 = Zf$; this f is given as $f = \int_0^1 Z_0(\cdot,\nu)\, d\nu$.

The mapping Z extends to an isometry from $L^2(\mathbb{R})$ into $L^2([0,1)^2)$, with corresponding extensions, in appropriate L^2-senses, of the properties A–F. In G we can allow $Z_0 \in L^2_{\text{loc}}(\mathbb{R}^2)$, the corresponding f being $\in L^2(\mathbb{R})$.

The mapping Z can also be extended to $L^1(\mathbb{R})$ through formula (2.4) in which the right-hand side converges absolutely for a.e. $t \in \mathbb{R}$. Doing so we have that

$$\int_0^1 \int_0^1 |(Zf)(t,\nu)|\, dt\, d\nu \le \int_{-\infty}^{\infty} |f(t)|\, dt \ , \qquad f \in L^1(\mathbb{R}) \ . \tag{2.6}$$

Hence Z maps $L^1(\mathbb{R})$ into $L^1([0,1)^2)$; the function f^1, see the comment after (1.9)–(1.11), demonstrates that not every quasi-periodic $Z_0 \in L^1_{\text{loc}}(\mathbb{R}^2)$ occurs as a Zf with $f \in L^1(\mathbb{R})$.

A remarkable property of the Zak transform is the zero-phenomenon: when Z_0 is quasi-periodic, see (2.5), and continuous as a function of t, ν, then Z_0 has a zero in $[0,1)^2$. The elements of S_0 have continuous Zak transforms, and so these Zak transforms have a zero in $[0,1)^2$.

3 Preliminaries from Gabor frame theory

3.1 Basic Gabor analysis

We develop the properties of Gabor systems that we need in this paper. Let $a > 0$, $b > 0$, $g \in L^2(\mathbb{R})$. We denote by (g,a,b) the system of time-frequency shifted functions $g_{na,mb}$, $n,m \in \mathbb{Z}$, and we call this a Gabor system. We call (g,a,b) a Gabor frame when there is $A > 0$ and $B < \infty$ (lower and upper frame bound, respectively) such that for all $f \in L^2(\mathbb{R})$

$$A\, \|f\|^2 \le \sum_{n,m} |(f, g_{na,mb})|^2 \le B\, \|f\|^2 \ . \tag{3.1}$$

We call (g,a,b) a Riesz-Gabor basic sequence when there is $C > 0$, $D < \infty$ such that for all $\underline{d} = (d_{nm})_{n,m\in\mathbb{Z}} \in l^2(\mathbb{Z}^2)$

$$C\, \|\underline{d}\|^2 \le \Big\| \sum_{n,m} d_{nm}\, g_{na,mb} \Big\|^2 \le D\, \|\underline{d}\|^2 \ . \tag{3.2}$$

When (g,a,b) is a Gabor system with a finite upper frame bound B, the operator S, defined by

$$f \in L^2(\mathbb{R}) \mapsto Sf = \sum_{n,m} (f, g_{na,mb})\, g_{na,mb} \in L^2(\mathbb{R}) \tag{3.3}$$

is bounded and positive semi-definite. This S commutes with all time-frequency shift operators involved in the series on the right-hand side of (3.3). When

(g, a, b) is a Gabor frame we have that S is positive definite (and hence invertible), and conversely.

Assume that (g, a, b) is a Gabor frame. Since S, and hence S^{-1}, commutes with all relevant time-frequency shift operators, we have for all $f \in L^2(\mathbb{R})$,

$$f = S(S^{-1}f) = \sum_{n,m} (f, {}^{\circ}\gamma_{na,mb})\, g_{na,mb} \ , \tag{3.4}$$

where ${}^{\circ}\gamma := S^{-1}g$. This ${}^{\circ}\gamma$ is called the canonical dual window associated with the Gabor frame (g, a, b). The system $({}^{\circ}\gamma, a, b)$ is also a Gabor frame, with frame operator S^{-1}.

There is the following result, see [15]. Assume that $g \in S_0$ and that $a > 0$, $b > 0$. Then the Gabor system (g, a, b) has a finite upper frame bound. When, moreover, (g, a, b) is a Gabor frame, we have that ${}^{\circ}\gamma = S^{-1}g \in S_0$.

We shall consider in this paper, in particular, the critical case $a = b = 1$. This case is called critical because of the following result of which a quick proof can be found at the author's guest page on www.univie.ac.at/NuHAG/. Let $g \in L^2(\mathbb{R})$, $a > 0$, $b > 0$.

Density theorem for Gabor systems

(i) (g, a, b) is a Gabor frame $\Rightarrow ab \le 1$,

(ii) (g, a, b) is a Riesz-Gabor basic sequence $\Rightarrow ab \ge 1$.

When $ab < 1$, there do exist well-behaved windows g such that (g, a, b) is a Gabor frame. This is so for any $a > 0$, $b > 0$ with $ab < 1$ when g is a Gaussian, see [24] and [25], or when g is a smooth function supported by an interval of length $\le 1/b$ while g is constant $\ne 0$ on an interval of length $\ge a$, see [7].

We now turn to the critical case $a = b = 1$. There do exist $g \in L^2(\mathbb{R})$ such that $(g, a = 1, b = 1)$ is a Gabor frame, but these g's are not simultaneously smooth and rapidly decaying. Examples are $g = \chi_{[0,1)}$, $g = \operatorname{sinc} \pi t$, while one of the best-behaved g's of this kind is perhaps $g = f^{1/2}$, see Fig. 1 and its properties noted in Sec. 1 below (1.9)–(1.11). We refer to [5] for a comprehensive survey of this phenomenon in connection with the Balian-Low theorem. In particular, $(g, 1, 1)$ is not a Gabor frame when $g \in S_0$.

The Zak transform plays a key role in Gabor analysis for the critical case $a = b = 1$. Assume that $g \in L^2(\mathbb{R})$ is such that $(g, 1, 1)$ has a finite upper frame bound B. Then we have $|Zg|^2 \le B$ a.e., and for $f \in L^2(\mathbb{R})$,

$$Z(Sf) = |Zg|^2\, Zf \tag{3.5}$$

in $L^2([0,1)^2)$-sense, see [14], Ch. 8. Here S is the frame operator, see (3.3). More generally, when φ is a continuous function on $\sigma(S) \subset [0, B]$ ($\sigma(S)$ denotes the spectrum of S), we have that for $f \in L^2(\mathbb{R})$,

$$Z(\varphi(S)f) = \varphi(|Zg|^2)\, Zf \tag{3.6}$$

in $L^2([0,1)^2)$-sense, see [22], Sec. 1.

It follows from (3.5) and the fact that Z is an isometry from $L^2(\mathbb{R})$ onto $L^2([0,1)^2)$ that $(g,1,1)$ is a Gabor frame with frame bounds $A>0$, $B<\infty$, if and only if $A \le |Zg|^2 \le B$, a.e. Clearly, $(g,1,1)$ is not a frame when $g \in S_0$: for then Zg is continuous and thus has a zero so that $A \le |Zg|^2$ a.e. cannot hold with an $A>0$.

4 Proof of the main result

In this section we shall prove the following result.

Theorem 4.1 Assume that $g^{(1)}, g^{(2)} \in S_0$ and that $Zg^{(1)}$ and $Zg^{(2)}$ have no common zeros. Let $f \in L^2(\mathbb{R})$. Then $f \in S_0$ if and only if $Zf(Zg^{(1)})^*$ and $Zf(Zg^{(2)})^*$ have absolutely convergent Fourier series.

The proof of this result requires some preparation.

Lemma 4.2 Let $g \in S_0$, $a>0$, $b>0$, and assume that (g,a,b) is a Gabor frame. Then

$$f \in S_0 \Leftrightarrow \sum_{n,m} |(f, g_{na,mb})| < \infty \ . \tag{4.1}$$

Proof $\Rightarrow$. Follows at once from (2.3).

$\Leftarrow$. With ${}^\circ\gamma = S^{-1}g$, where S is the frame operator corresponding to (g,a,b), we have that $({}^\circ\gamma, a, b)$ is a Gabor frame with frame operator S^{-1} and canonical dual $(S^{-1})^{-1}\,{}^\circ\gamma = g$. Also, ${}^\circ\gamma \in S_0$ by [15]. Let $f \in L^2(\mathbb{R})$. Then f has the $L^2(\mathbb{R})$-convergent expansion

$$f = \sum_{n,m} (f, g_{na,mb})\, {}^\circ\gamma_{na,mb} \ . \tag{4.2}$$

Hence, when $\Sigma_{n,m} |(f, g_{na,mb})| < \infty$, the right-hand side of (4.2) converges absolutely in S_0 since $\| \ \|_{S_0}$ is shift-invariant. Since S_0 is a Banach space, we have then that $f \in S_0$.

Lemma 4.3 Let $g \in S_0$ and assume that $(g, a=\frac{1}{2}, b=1)$ is a Gabor frame. Let $f \in L^2(\mathbb{R})$. Then $f \in S_0$ if and only if

$$Zf(Zg)^* \quad \text{and} \quad (t,\nu) \mapsto (Zf)(t,\nu)(Zg)^*(t-\tfrac{1}{2},\nu) \tag{4.3}$$

have absolutely convergent Fourier series.

Proof We note that for $t, \nu \in \mathbb{R}$ and $n,m \in \mathbb{Z}$

$$(Zg_{1/2,0})(t,\nu) = (Zg)(t-\tfrac{1}{2},\nu) \ , \qquad (g_{1/2,0})_{n,m} = g_{n+1/2,m} \ . \tag{4.4}$$

Hence by 2.2.E, the $(n,m)^{\text{th}}$ Fourier coefficient of the two functions in (4.3) are given by

$$(f, g_{-m,n}) \quad \text{and} \quad (f, g_{-m+1/2,n}) \ , \qquad n,m \in \mathbb{Z} \ , \tag{4.5}$$

respectively. When $f \in S_0$ the two sequences in (4.5) are absolutely summable by (2.3). Conversely, when the two sequences in (4.5) are absolutely summable, we have that $\sum_{n,m} |(f, g_{n/2,m})| < \infty$, and thus $f \in S_0$ by Lemma 4.2.

Proof of Theorem 4.1 Assume that $f \in S_0$. Then (2.3) and 2.2.E show that $Zf(Zg^{(1)})^*$ and $Zf(Zg^{(2)})^*$ have absolutely convergent Fourier series.

For the converse, we consider the "multi-window" operator S, defined for $k \in L^2(\mathbb{R})$ by

$$Sk = S^{(1)}k + S^{(2)}k = \sum_{n,m} (k, g_{nm}^{(1)})\, g_{nm}^{(1)} + \sum_{n,m} (k, g_{nm}^{(2)})\, g_{nm}^{(2)} \ , \tag{4.6}$$

also see [14], Comment 1 on pp. 158–159. By (3.5), applied to $S^{(1)}$ and $S^{(2)}$, we have

$$Z(Sk) = (|Zg^{(1)}|^2 + |Zg^{(2)}|^2)\, Zk \ , \qquad k \in L^2(\mathbb{R}) \ . \tag{4.7}$$

By assumption, $|Zg^{(1)}|^2 + |Zg^{(2)}|^2$ is strictly positive, $Zg^{(1)}$ and $Zg^{(2)}$ being continuous functions with no common zeros. Hence S is boundedly invertible on $L^2(\mathbb{R})$ since Z is an isometry from $L^2(\mathbb{R})$ onto $L^2([0,1)^2)$. Applying (4.7) we have for $g \in L^2(\mathbb{R})$ that

$$Z(S^{-1}g) = \frac{Zg}{|Zg^{(1)}|^2 + |Zg^{(2)}|^2} \ . \tag{4.8}$$

We shall show now that $\gamma^{(i)} := S^{-1}g^{(i)} \in S_0$, $i = 1,2$. To that end, we let $l \in S_0$ such that $(l, \frac{1}{2}, 1)$ is a Gabor frame. Since $g^{(i)} \in S_0$ we see that $|Zg^{(i)}|^2$ has an absolutely convergent Fourier series, $i = 1,2$. By strict positivity of $|Zg^{(1)}|^2 + |Zg^{(2)}|^2$ we then have by Wiener's lemma, see [14], Sec. 13.3, that $(|Zg^{(1)}|^2 + |Zg^{(2)}|^2)^{-1}$ has an absolutely convergent Fourier series. Also, $Zg^{(i)}(Zl)^*$ and $Zg^{(i)}(Zl_{1/2,0})^*$ have absolutely convergent Fourier series since all involved windows are in S_0, $i = 1,2$. Combining all this and using (4.8), we see that $Z\gamma^{(i)}(Zl)^*$ and $Z\gamma^{(i)}(Zl_{1/2,0})^*$ have absolutely convergent Fourier series, whence $\gamma^{(i)} \in S_0$ by Lemma 4.3, $i = 1,2$.

Now assume that $f \in L^2(\mathbb{R})$ is such that $Zf(Zg^{(i)})^*$ have absolutely convergent Fourier series, i.e., $\sum_{n,m} |(f, g_{n,m}^{(i)})| < \infty$, $i = 1,2$. Then

$$\begin{aligned} f = S^{-1}(Sf) \ &= \ \sum_{n,m} (f, g_{nm}^{(1)})\, S^{-1}\, g_{nm}^{(1)} + \sum_{n,m} (f, g_{nm}^{(2)})\, S^{-1}\, g_{nm}^{(2)} \\ &= \ \sum_{n,m} (f, g_{nm}^{(1)})\, \gamma_{nm}^{(1)} + \sum_{n,m} (f, g_{nm}^{(2)})\, \gamma_{nm}^{(2)} \ , \end{aligned} \tag{4.9}$$

where it has been used that S, see (4.6), and whence S^{-1} commutes with all time-frequency shift operators occurring in (4.9). Therefore, $f \in S_0$ since S_0 is a time-frequency shift invariant Banach space with $\|\gamma_{nm}^{(i)}\|_{S_0} = \|\gamma^{(i)}\|_{S_0} < \infty$, $i = 1, 2$.

5 Application to Gaussian window Gabor system at critical density

We let $h(t)$ be the standard Gaussian $2^{1/4}\exp(-\pi t^2)$, and we consider the critical-density Gabor system

$$(h, a = 1, b = 1) = (h_{nm})_{n,m \in \mathbb{Z}} \tag{5.1}$$

with frame operator S, see (3.3). We are interested in properties of the windows $f^\lambda := S^{-\lambda}h$, $\lambda \geq 0$, in order to illustrate our main result in Sec. 4 and to generalize what is known for the cases $\lambda = 1, \frac{1}{2}$ (see [4], [16]–[18] and [21]). One of the findings is that the negative conclusion of the Balian-Low theorem just holds true in S_0.

The definition of $f^\lambda = S^{-\lambda}h$ is awkward for $\lambda > 0$ since the frame operator S is not invertible (because h is in S_0). We circumvent this problem by defining f^λ indirectly via the Zak transform domain. We compute the Zak transform of h as

$$\begin{aligned}(Zh)(t,\nu) &= 2^{1/4} \sum_{k=-\infty}^{\infty} e^{-\pi(t-k)^2}\, e^{2\pi i k\nu} \\ &= 2^{1/4}\, e^{-\pi t^2}\, \vartheta_3(\pi(\nu - it), e^{-\pi}) \\ &= -2^{1/4}\, i\, e^{-\pi(t-1/2)^2 + \pi i(\nu-1/2)^2}\, \vartheta_1(\pi(\nu - \tfrac{1}{2} - i(t - \tfrac{1}{2}), e^{-\pi})\ ,\end{aligned} \tag{5.2}$$

where

$$\vartheta_3(z,q) = \sum_{k=-\infty}^{\infty} q^{k^2}\, e^{2ikz}\ ,\quad \vartheta_1(z,q) = -i \sum_{k=-\infty}^{\infty} (-1)^k\, q^{(k+1/2)^2}\, e^{(2k+1)iz} \tag{5.3}$$

are theta functions as in [27], Ch. 21. It follows from the properties of the theta functions that Zh vanishes in $[0,1)^2$ at $(t,\nu) = (\frac{1}{2}, \frac{1}{2})$ and nowhere else. Furthermore,

$$(Zh)(t,\nu) = -2^{1/4}\, \pi\, \vartheta_1'(0)(t - \tfrac{1}{2} + i(\nu - \tfrac{1}{2})) + O((t-\tfrac{1}{2})^2 + (\nu - \tfrac{1}{2})^2)\ , \tag{5.4}$$

where

$$\vartheta_1'(0) = \frac{d}{dz}\,\vartheta_1(z, q = e^{-\pi})(z = 0) = 2\sum_{k=0}^{\infty} (-1)^k\, e^{-\pi(k+1/2)^2} \neq 0\ . \qquad (5.5)$$

A second observation is that we have by functional calculus in the Zak transform domain, see (3.6), that for $\lambda \geq 0$

$$Z(S^\lambda f) = |Zh|^{2\lambda}\, Zf\ , \qquad f \in L^2(\mathbb{R})\ . \qquad (5.6)$$

Combining this with the inversion formula 2.2.D for the Zak transform, we are led to the following definition.

Definition 5.1 Let $\lambda \geq 0$. For $t \in \mathbb{R}$, not of the form $n + \frac{1}{2}$ with integer n, we set

$$f^\lambda(t) := \int_0^1 (Zh)(t,\nu)/|(Zh)(t,\nu)|^{2\nu}\, d\nu\ . \qquad (5.7)$$

Definition 5.1 is such that we formally have $Zf^\lambda = Zh/|Zh|^{2\lambda}$, see (5.6) and take $f = S^{-\lambda}h$ (formally). Note that, from what has been said about Zh before,

$$0 \leq \lambda < \tfrac{3}{2} \quad\Rightarrow\quad Zh/|Zh|^{2\lambda} \in L^1([0,1)^2)\ , \qquad (5.8)$$

$$0 \leq \lambda < 1 \quad\Rightarrow\quad Zh/|Zh|^{2\lambda} \in L^2([0,1)^2)\ , \qquad (5.9)$$

$$0 \leq \lambda \leq \tfrac{1}{2} \quad\Rightarrow\quad Zh/|Zh|^{2\lambda} \in L^\infty([0,1)^2)\ . \qquad (5.10)$$

More precisely, when $\lambda \geq 0$ we have that $Zh/|Zh|^{2\lambda}$ is in any $L^p([0,1)^2)$ with $(\lambda - \frac{1}{2})\,p < 1$ (here we use $0 \cdot \infty = 0$ for the case $\lambda = \frac{1}{2}$).

Theorem 5.2 There holds

$$0 \leq \lambda < \tfrac{3}{2} \quad\Rightarrow\quad f^\lambda \in L^1_{\rm loc}(\mathbb{R})\ , \qquad (5.11)$$

$$0 \leq \lambda < 1 \quad\Rightarrow\quad f^\lambda \in L^2 \cap L^\infty \cap C(\mathbb{R})\ . \qquad (5.12)$$

Proof The statement in (5.11) follows from the definition in (5.7), quasi-periodicity of Zh (see 2.2.B) and Fubini's theorem.

To prove (5.12), we fix $0 \leq \lambda < 1$. Then $Zh/|Zh|^{2\lambda} \in L^2([0,1)^2)$ by (5.9). Since $Zh/|Zh|^{2\lambda}$ is quasi-periodic, we get from the discussion following 2.2.G that there is a unique $f \in L^2(\mathbb{R})$ such that $Zf = Zh/|Zh|^{2\lambda}$ a.e. By the L^2-extension of the inversion formula 2.2.D we have that this f agrees in L^2-sense with f^λ. The statement that $f^\lambda \in L^\infty \cap C(\mathbb{R})$ follows from the definition in (5.7), the fact that $Zh/|Zh|^{2\lambda}$ is smooth on $[0,1)^2$ except at $(t,\nu) = (\frac{1}{2}, \frac{1}{2})$, formula

(5.4), the fact that $|\nu - \frac{1}{2}|^{1-2\lambda} \in L^1([0,1))$, and dominated convergence.

The case that $\lambda = 1$ was excluded in Thm. 5.2. We have that

$$f^1(t) = -\frac{2^{3/4}}{\vartheta_1'(0)}\, e^{\pi t^2} \sum_{n-1/2 \geq |t|} (-1)^n\, e^{-\pi(n-1/2)^2} \tag{5.13}$$

when t is not a half-integer, see [4], [16]–[18]. In [18], 4.4 Theorem, it is noted that $f^1 \in L^\infty(\mathbb{R})\backslash L^p(\mathbb{R})$ for any p with $1 \leq p < \infty$. Accordingly, one should consider Zf^1 in distributional sense. This is done in [18], Sec. 4.4, where it is shown that $Zf^1(Zh)^* = 1$ in distributional sense. Hence, in the main result it is certainly necessary to consider more than one "test" window g: $Zf^1(Zh)^*$ has an absolutely convergent Fourier series while f^1 is not even in $L^2(\mathbb{R})$. It is also shown in [18], Sec. 4.4, that f^1 is Fourier invariant (in distributional sense). Also see Fig. 1, which shows discontinuities of f^1 at all half-integers.

The case $\lambda = 1$ is furthermore interesting since f^1 is formally the canonical dual $S^{-1}h$ corresponding to the Gabor "frame" $(h,1,1)$, and satisfies $(f^1, h_{nm}) = \delta_{no}\delta_{mo}$. In fact, this f^1 was constructed in [16], 2.14, as the unique regular distribution satisfying $(f^1, h_{nm}) = \delta_{no}\delta_{mo}$.

A related result can be found in [18], 4.8. Here an explicit formula for the unique coefficients $c_{nm}(a,b)$ with $c_{nm}(a,b) \to 0$, $n^2+m^2 \to \infty$, in the Gabor expansion of the time-frequency shifted window $h_{a,b}$ with $a, b \in \mathbb{R}$ is given. There holds, in distributional sense,

$$h_{a,b} = \sum_{n,m} c_{nm}(a,b)\, h_{nm}\ , \tag{5.14}$$

where

$$c_{nm}(a,b) = (h_{a,b}, f^1) = \frac{(-1)^{n+m}\vartheta_1(\pi(a+ib))}{\pi\vartheta_1'(0)(a+ib-n-im)}\, e^{2\pi iab - \pi b^2}\ . \tag{5.15}$$

Note that $|c_{nm}(a,b)|$ exhibits a $(n^2+m^2)^{-1/2}$ decay, and is, therefore, not in $l^2(\mathbb{Z}^2)$ unless a and b are integer, in which case $\vartheta_1(\pi(a+ib)) = 0$.

Proposition 5.3 Let $1 < \lambda < \frac{3}{2}$. Then $f^\lambda \notin L^\infty(\mathbb{R})$, $f^\lambda \notin L^1(\mathbb{R})$.

Proof To show that $f^\lambda \notin L^\infty(\mathbb{R})$, we consider $f_\lambda(t)$ near $t = \frac{1}{2}$, and to that end we use (5.4). Note that

$$\left| \int_0^1 \frac{t - \frac{1}{2} + i(\nu - \frac{1}{2})}{|t - \frac{1}{2} + i(\nu - \frac{1}{2})|^{2\lambda}}\, d\nu \right| = \frac{\Gamma(\lambda - \frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\lambda)}\, |t - \tfrac{1}{2}|^{2-2\lambda} + O(|t - \tfrac{1}{2}|)\ , \tag{5.16}$$

hence $|f^\lambda(t)|$ has leading order behaviour $|t - \frac{1}{2}|^{2-2\lambda}$ as $t \to \frac{1}{2}$. Therefore, $f^\lambda \notin L^\infty(\mathbb{R})$ since $\lambda > 1$.

Next suppose that $f^\lambda \in L^1(\mathbb{R})$. By Fourier invariance of h we have, see 2.2.F,

$$(Zh)(t,\nu) = e^{2\pi i\nu t}(Zh)(-\nu,t) \ . \tag{5.17}$$

Since $f^\lambda \in L^1(\mathbb{R})$ we have $Zf^\lambda \in L^1([0,1)^2)$, see (2.6), and by the L^1-extension of 2.2.D there holds a.e. that

$$(Zf^\lambda)(t,\nu) = \sum_k f^\lambda(t-k)\, e^{2\pi ik\nu} = (Zh)(t,\nu)/|(Zh)(t,\nu)|^{2\lambda} \ . \tag{5.18}$$

Because of (5.17) we have then that

$$e^{-2\pi i\nu t}(Zf^\lambda)(t,\nu) = (Zf^\lambda)(-\nu,t) \ , \qquad \text{a.e. } t,\nu \ . \tag{5.19}$$

Now integrate this identity over $t \in [0,1)$. For the left-hand side we get

$$\int_0^1 e^{-2\pi i\nu t}(Zf^\lambda)(t,\nu)\, dt = \int_0^1 \sum_k f^\lambda(t-k)\, e^{-2\pi i(t-k)\nu}\, dt$$

$$= \int_{-\infty}^{\infty} f^\lambda(t)\, e^{-2\pi it\nu}\, dt = (\mathcal{F}f^\lambda)(\nu) \ , \qquad \text{a.e. } \nu \ , \tag{5.20}$$

while for the right-hand side we get by the L^1-extension of 2.2.D that

$$\int_0^1 (Zf^\lambda)(-\nu,t)\, dt = f^\lambda(-\nu) \ , \qquad \text{a.e. } \nu \ . \tag{5.21}$$

Hence $f^\lambda(-\nu) = (\mathcal{F}f^\lambda)(\nu)$, a.e. ν. But since $f^\lambda \in L^1(\mathbb{R})$, we have that $\mathcal{F}f^\lambda \in L^\infty(\mathbb{R})$, whence $f^\lambda \in L^\infty(\mathbb{R})$. Contradiction.

Note 5.4 We also have $\lambda \geq \frac{3}{2} \Rightarrow f^\lambda \notin L^1_{\text{loc}}(\mathbb{R})$.

We limit our attention in the sequel to the case that $0 < \lambda < 1$.

Theorem 5.5 Let $0 < \lambda < 1$. Then $Zf^\lambda = Zh/|Zh|^{2\lambda}$, and

(i) $\mathcal{F}f^\lambda = f^\lambda$,

(ii) f^λ and $f^{1-\lambda}$ are dual in the sense that $Zf^\lambda(Zf^{1-\lambda})^* = 1$.

Proof The proof of (i) follows the same argument that was used in the proof of Prop. 5.3 to show (5.21); here we also use that f^λ is even (an $f \in L^2(\mathbb{R})$ is even if and only if $(Zf)(t,\nu) = (Zf)(-t,-\nu)$, whence evenness of f^λ is inherited from h). The proof of (ii) follows from the fact, as in the proof of (5.18), that

$Zf^\lambda = Zh/|Zh|^{2\lambda}$, $Zf^{1-\lambda} = Zh/|Zh|^{2-2\lambda}$, a.e.

Theorem 5.6 Let $0 < \lambda < 1$. Then $f^\lambda \in L^1(\mathbb{R})$.

Proof We follow to a large extent the proof of [21], Thm. 2 in Sec. 4, that would apply to the case $\lambda = \frac{1}{2}$. We have for $t \in [0,1)$, $t \neq \frac{1}{2}$ and $n \in \mathbb{Z}$ by partial integration that

$$f^\lambda(t+n) = \int_0^1 e^{2\pi i n\nu} (Zf^\lambda)(t,\nu)\,d\nu = \frac{-1}{2\pi i n} \int_0^1 e^{2\pi i n \nu} \frac{\partial Z f^\lambda}{\partial \nu}(t,\nu)\,d\nu\ . \tag{5.22}$$

Let $1 < p \leq 2$, $q = (1-p^{-1})^{-1}$. By Hölder's inequality

$$\begin{aligned} \sum_{n\neq 0} |f^\lambda(t+n)| &\leq \left(\sum_{n\neq 0} \left|\frac{1}{2\pi n}\right|^p\right)^{1/p} \left(\sum_{n\neq 0} \left|\int_0^1 e^{2\pi i n\nu} \frac{\partial Zf^\lambda}{\partial\nu}(t,\nu)\,d\nu\right|^q\right)^{1/q} \\ &\leq \frac{1}{2\pi}(2\zeta(p))^{1/p} \left\| \left(\int_0^1 e^{2\pi i n\nu} \frac{\partial Zf^\lambda}{\partial \nu}(t,\nu)\,d\nu\right)_{n\in\mathbb{Z}} \right\|_{l^q(\mathbb{Z})} . \end{aligned} \tag{5.23}$$

By the Hausdorff-Young inequality, see [8], 13.5.1 on p. 153 ($p' = q$), we have for $t \in [0,1)$, $t \neq \frac{1}{2}$, that

$$\left\| \left(\int_0^1 e^{2\pi i n\nu} \frac{\partial Zf^\lambda}{\partial\nu}(t,\nu)\,d\nu\right)_{n\in\mathbb{Z}} \right\|_{l^q(\mathbb{Z})} \leq \left\| \frac{\partial Zf^\lambda}{\partial \nu}(t,\cdot)\right\|_{L^p([0,1))} . \tag{5.24}$$

We compute for $t \neq \frac{1}{2}$, $t \in [0,1)$ that

$$\begin{aligned} \left|\frac{\partial Zf^\lambda}{\partial\nu}\right| = \left|\frac{\partial}{\partial\nu}\frac{Zh}{|Zh|^{2\lambda}}\right| &= \left| \frac{\frac{\partial}{\partial\nu}Zh}{|Zh|^{2\lambda}} - 2\lambda\frac{Zh}{|Zh|^{2\lambda+1}}\frac{\partial}{\partial\nu}|Zh| \right| \\ &\leq \frac{(2\lambda+1)\left\|\frac{\partial}{\partial\nu}Zh\right\|_\infty}{|Zh|^{2\lambda}} . \end{aligned} \tag{5.25}$$

From the analysis given on Zh earlier in this section we infer the existence of $\varepsilon > 0$, $\delta > 0$, $C > 0$, $D > 0$, $c > 0$ such that

$$|(Zh)(t,\nu)| \geq (C^2(t-\tfrac{1}{2})^2 + D^2(\nu - \tfrac{1}{2})^2)^{1/2} \tag{5.26}$$

when $|t-\frac{1}{2}| \le \varepsilon$ and $|\nu-\frac{1}{2}| \le \delta$, and $|(Zh)(t,\nu)| \ge c$ otherwise. Therefore,

$$\Big|\frac{\partial Zf^{\lambda}}{\partial \nu}(t,\nu)\Big| = O((C^2(t-\tfrac{1}{2})^2 + D^2(\nu-\tfrac{1}{2})^2)^{-\lambda}) \tag{5.27}$$

when $|t-\frac{1}{2}| \le \varepsilon$ and $|\nu-\frac{1}{2}| \le \delta$, and $|\frac{\partial Zf^{\lambda}}{\partial \nu}(t,\nu)| = O(1)$ otherwise. We thus get that

$$\Big\|\frac{\partial Zf^{\lambda}}{\partial \nu}(t,\cdot)\Big\|_p^p = \int_0^1 \Big|\frac{\partial Zf^{\lambda}}{\partial \nu}(t,\nu)\Big|^p d\nu$$

$$= \begin{cases} O(1) + O\left(\displaystyle\int_{1/2-\delta}^{1/2+\delta} (C^2(t-\tfrac{1}{2})^2 + D^2(\nu-\tfrac{1}{2})^2)^{-p\lambda}\, d\nu\right) & , \quad |t-\frac{1}{2}| \le \varepsilon \\ O(1) & , \quad |t-\frac{1}{2}| > \varepsilon\ . \end{cases} \tag{5.28}$$

We compute furthermore that

$$\int_{1/2-\delta}^{1/2+\delta} (C^2(t-\tfrac{1}{2})^2 + D^2(\nu-\tfrac{1}{2})^2)^{-p\lambda}\, d\nu = 2C^{1-2p\lambda}\, D^{-1}|t-\tfrac{1}{2}|^{1-2p\lambda} \int_0^{x(t)} \frac{dx}{(1+x^2)^{p\lambda}}\ , \tag{5.29}$$

where $x(t) = D\delta/C|t-\frac{1}{2}|$.

Case $1/2 \le \lambda < 1$ Then, with p as before, $2p\lambda > 1$ so that $\int_0^{\infty} (1+x^2)^{-p\lambda}\, dx < \infty$. Thus we find on combining (5.23), (5.24), (5.28) and (5.29) that

$$\sum_{n\neq 0} |f^{\lambda}(t+n)| = O(|t-\tfrac{1}{2}|^{p^{-1}-2\lambda})\ , \qquad t \in [0,1)\ . \tag{5.30}$$

The right-hand side of (5.30) is integrable over $[0,1)$ when $p^{-1} - 2\lambda > -1$, i.e. when $p \in (1, (2\lambda-1)^{-1}) \neq \emptyset$. Hence $f^{\lambda} \in L^1(\mathbb{R})$.

Case $0 < \lambda < \frac{1}{2}$ Now take $p > 1$ such that $2p\lambda < 1$. Then

$$\int_0^{x(t)} \frac{dx}{(1+x^2)^{p\lambda}} = O((x(t))^{1-2p\lambda}) = O(|t-\tfrac{1}{2}|^{2p\lambda-1})\ . \tag{5.31}$$

Hence $\sum_{n\neq 0} |f^{\lambda}(t+n)| = O(1)$, $t \in [0,1)$, so that $f^{\lambda} \in L^1(\mathbb{R})$.

Note 5.7 According to [21], Thm. 1, we have $S_a^{-1/2}h \to f^{1/2}$ as $a \uparrow 1$, where S_a is the frame operator corresponding to the Gabor frame (h,a,a). In the proof it is used that $\|S_a^{-1/2}h\| = a \to 1 = \|f^{1/2}\|$ as $a \uparrow 1$. Hence it is not obvious how to prove [21], Thm. 1 for general $\lambda \in (0,1)$.

We shall now apply our main result to find out when $f^\lambda \in S_0$ with $\lambda \in (0,1)$. We use for this the following result.

Proposition 5.8 Let $g \in \mathcal{S}$ be such that $(Zg)(\frac{1}{2},\frac{1}{2}) \neq 0$, and let $0 < \lambda < 1$. Denote by $c_{nm}(\lambda) = (f^\lambda, g_{-m,n})$ the $(n,m)^{\text{th}}$ Fourier coefficient of $Zf^\lambda(Zg)^*$, see 2.2.E. Then there is a $C \neq 0$ such that for integer n, m with $n^2+m^2 \to \infty$,

$$(-1)^{n+m}\, c_{nm}(\lambda)\, r^{3-2\lambda}\, e^{-i\varphi} \to C \ , \tag{5.32}$$

where we have written $n + im = r\,e^{i\varphi}$ with $r > 0$, $\varphi \in \mathbb{R}$.

Proof Since $Zf^\lambda(Zg)^*$ is $(1,1)$-periodic and in $C^\infty([0,1)^2/\{(\frac{1}{2},\frac{1}{2})\})$, the behaviour of its Fourier coefficients

$$c_{nm}(\lambda) = \int_0^1\int_0^1 (Zf^\lambda)(t,\nu)(Zg)^*(t,\nu)\, e^{-2\pi int-2\pi im\nu}\, dt\, d\nu \tag{5.33}$$

when $n^2+m^2 \to \infty$ is in leading order determined by the behaviour of $Zf^\lambda(Zg)^*$ at $(t,\nu) = (\frac{1}{2},\frac{1}{2})$. Note that by (5.4) we have

$$(Zf^\lambda)(t,\nu)(Zg)^*(t,\nu) = \tag{5.34}$$

$$D\,\frac{t-\frac{1}{2}+i(\nu-\frac{1}{2})}{((t-\frac{1}{2})^2+(\nu-\frac{1}{2})^2)^\lambda} + O(((t-\tfrac{1}{2})^2+(\nu-\tfrac{1}{2})^2)^{-\lambda+1})$$

as $(t,\nu) \to (\frac{1}{2},\frac{1}{2})$, where

$$D = -(2^{1/4}\,\pi\,\vartheta_1'(0))^{1-2\lambda}(Zg)^*(\tfrac{1}{2},\tfrac{1}{2}) \neq 0\ . \tag{5.35}$$

Therefore, the leading order behaviour of $c_{nm}(\lambda)$ as $n^2+m^2 \to \infty$ coincides with that of

$$d_{nm}(\lambda) = D\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{t-\frac{1}{2}+i(\nu-\frac{1}{2})}{((t-\frac{1}{2})^2+(\nu-\frac{1}{2})^2)^\lambda}\,K(t,\nu)\,e^{-2\pi int-2\pi im\nu}\,dt\,d\nu\ , \tag{5.36}$$

where K is any smooth and rapidly decaying element of $C^\infty(\mathbb{R}^2)$ with $K(\frac{1}{2},\frac{1}{2}) = 1$. For the computations below it will be convenient to choose

$$K(t,\nu) = \exp(-\pi(t-\tfrac{1}{2})^2 - \pi(\nu-\tfrac{1}{2})^2)\ . \tag{5.37}$$

We thus compute, using polar coordinates $t-\frac{1}{2}+i(\nu-\frac{1}{2})=\rho\,e^{i\vartheta}$ and $n+im=r\,e^{i\varphi}$ with $0\le\rho,r<\infty$, $0\le\vartheta,\varphi<2\pi$,

$$d_{nm}(\lambda) \;=$$

$$= \;(-1)^{n+m}\,D\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{t-\frac{1}{2}+i(\nu-\frac{1}{2})}{((t-\frac{1}{2})^2+(\nu-\frac{1}{2})^2)^{\lambda}}\,e^{-2\pi in(t-1/2)-2\pi im(\nu-1/2)}\;\cdot$$

$$\cdot\; e^{-\pi(t-1/2)^2-\pi(\nu-1/2)^2}\,dt\,d\nu \;=$$

$$= \;(-1)^{n+m}\,D\int\limits_{0}^{2\pi}\int\limits_{0}^{\infty}\frac{\rho\,e^{i\vartheta}}{\rho^{2\lambda}}\,e^{-2\pi i\rho r\cos(\vartheta-\varphi)}\,e^{-\pi\rho^2}\,\rho\,d\rho\,d\vartheta\;. \tag{5.38}$$

On simplifying and by using that (see [1], 9.1.42–43 on p. 361)

$$\int\limits_{0}^{2\pi} e^{i\vartheta}\,e^{-2\pi i\rho r\cos(\vartheta-\varphi)}\,d\vartheta = -2\pi i\,J_1(2\pi\rho r)\,e^{i\varphi}\;, \tag{5.39}$$

where J_1 is the Bessel function of the first kind and of order 1, we obtain

$$d_{nm}(\lambda) = -2\pi i(-1)^{n+m}\,D\,e^{i\varphi}\int\limits_{0}^{\infty}\rho^{2-2\lambda}\,e^{-\pi\rho^2}\,J_1(2\pi\rho r)\,d\rho\;. \tag{5.40}$$

The remaining integral in (5.40) can be expressed, by using [1], 11.4.28 on p. 486, as

$$\int\limits_{0}^{\infty}\rho^{2-2\lambda}\,e^{-\pi\rho^2}\,J_1(2\pi\rho r)\,d\rho = \tfrac{1}{2}\Gamma(2-\lambda)\,\pi^{\lambda-1}\,r\,M(2-\lambda,2,-\pi r^2)\;, \tag{5.41}$$

where $M(a,b,z)$ is Kummer's function ${}_1F_1(a\,;\,b\,;\,z)$ (confluent hypergeometric function) as in [1], 13.1.2 on p. 504. Using [1], 13.1.5 on p. 504, we have as $r\to\infty$

$$M(2-\lambda,2,-\pi r^2) = \frac{(\pi r^2)^{\lambda-2}}{\Gamma(\lambda)}\,(1+O(r^{-2}))\;, \tag{5.42}$$

so that, as $r\to\infty$,

$$\int\limits_{0}^{\infty}\rho^{2-2\lambda}\,e^{-\pi\rho^2}\,J_1(2\pi\rho r)\,d\rho = \frac{\Gamma(2-\lambda)}{2\Gamma(\lambda)}\,(\pi r)^{2\lambda-3}(1+O(r^{-2}))\;. \tag{5.43}$$

Therefore, the leading behaviour of $d_{nm}(\lambda)$, and, hence, that of $c_{nm}(\lambda)$, as $n^2 + m^2 \to \infty$ is given as

$$-\pi i(-1)^{n+m}\, D\, e^{i\varphi}\, \frac{\Gamma(2-\lambda)}{\Gamma(\lambda)}\, (\pi r)^{2\lambda-3}\ , \tag{5.44}$$

with D given in (5.35) and $n + im = r\, e^{i\varphi}$, as required.

Note 5.9 Compare with [23], Thms. 5.9–10 on pp. 122–123, where Fourier integrals as in (5.38) are considered.

Theorem 5.10 There holds

$$0 < \lambda < \tfrac{1}{2} \quad \Rightarrow \quad f^{\lambda} \in S_0\ , \tag{5.45}$$

$$\tfrac{1}{2} \le \lambda < 1 \quad \Rightarrow \quad f^{\lambda} \notin S_0\ . \tag{5.46}$$

Proof We take $g^{(1)}, g^{(2)} \in \mathcal{S}$ with $(Zg^{(1)})(\frac{1}{2},\frac{1}{2}) \neq 0 \neq (Zg^{(2)})(\frac{1}{2},\frac{1}{2})$ and such that $Zg^{(1)}$ and $Zg^{(2)}$ have no common zeros. We have that

$$\sum_{n,m\neq(0,0)} \frac{1}{(n^2+m^2)^{3/2-\lambda}} < \infty \Leftrightarrow \lambda < \tfrac{1}{2}\ , \tag{5.47}$$

and we get (5.45)–(5.46) at once from the main result.

Note 5.11 The proof of the leading order behaviour formula (5.44) for the $c_{nm}(\lambda)$ continues to work when $1 \le \lambda < \frac{3}{2}$. In the case that $\lambda = 1$ this yields for $(f^1, g_{-m,n})$ the leading order behaviour

$$(-1)^{n+m}\, \frac{i(Zg)^*(\frac{1}{2},\frac{1}{2})\, e^{i\varphi}}{2^{1/4}\, \pi\, \vartheta_1'(0)\, r} \tag{5.48}$$

when $n+im = r\, e^{i\varphi}$ with $r \to \infty$. Consequently, when $g \in \mathcal{S}$ and $(Zg)(\frac{1}{2},\frac{1}{2}) \neq 0$, the coefficients (g, f^1_{nm}) in the Gabor expansion (in distributional sense) of g,

$$g = \sum_{n,m} (g, f^1_{nm})\, h_{nm} \tag{5.49}$$

(Gaussian window h, critical case), have leading order behaviour

$$-\frac{(-1)^{n+m}(Zg)(\frac{1}{2},\frac{1}{2})}{2^{1/4}\, \pi\, \vartheta_1'(0)(n+im)}\ . \tag{5.50}$$

6 Extension of the main result

It was kindly observed to the author by K. Gröchenig that the main result of this paper has an extension to more general modulation spaces.

Taking the notation and definitions as in [14], Chs. 11–12, we let $v, m \geq 0$ be locally integrable on $\mathbb{R}^{2d}$, where v is assumed to be submultiplicative on $\mathbb{R}^{2d}$ and m is assumed to be v-moderate on $\mathbb{R}^{2d}$ so that

$$v(z_1+z_2) \leq v(z_1)\,v(z_2)\,; \quad m(z_1+z_2) \leq C\,v(z_1)\,m(z_2)\ , \qquad z_1, z_2 \in \mathbb{R}^{2d}\ , \tag{6.1}$$

where C is some constant > 0. Now let $1 \leq p, q \leq \infty$, and let $g^{(r)} \in M_v^1$, $r = 1, ..., N$, be such that the $Zg^{(r)}$ have no common zeros. Also, let $f \in L^2(\mathbb{R}^d)$. Then $f \in M_m^{p,q}$ if and only if $Zf(Zg^{(r)})^*$ have Fourier series

$$\sum_{k,l} c_{kl}^{(r)}\, e^{2\pi i k\cdot t + 2\pi i l\cdot \nu}\ , \tag{6.2}$$

where

$$(c_{l,-k}^{(r)})_{k\in\mathbb{Z}^d, l\in\mathbb{Z}^d} \in l_m^{p,q}(\mathbb{Z}^{2d})\ , \qquad r = 1, ..., N\ . \tag{6.3}$$

The proof contains essentially the same ingredients as the proof of the main result given in Sec. 4.

This extension can be used in conjunction with Prop. 5.8 in Sec. 5 to tell to which spaces a particular f^λ belongs. For instance, when $v_s(t,\nu) = (1 + (t^2 + \nu^2)^{1/2})^s$, then

$$f^\lambda \in M_{v_s}^1 \Leftrightarrow 2\lambda + s < 1\ . \tag{6.4}$$

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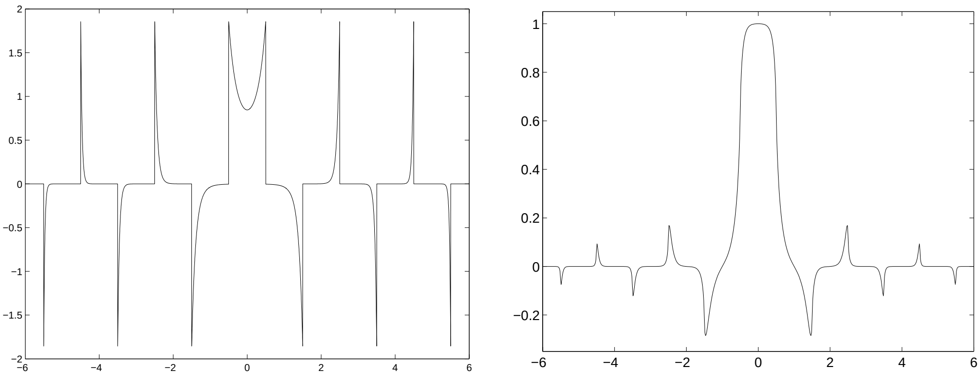
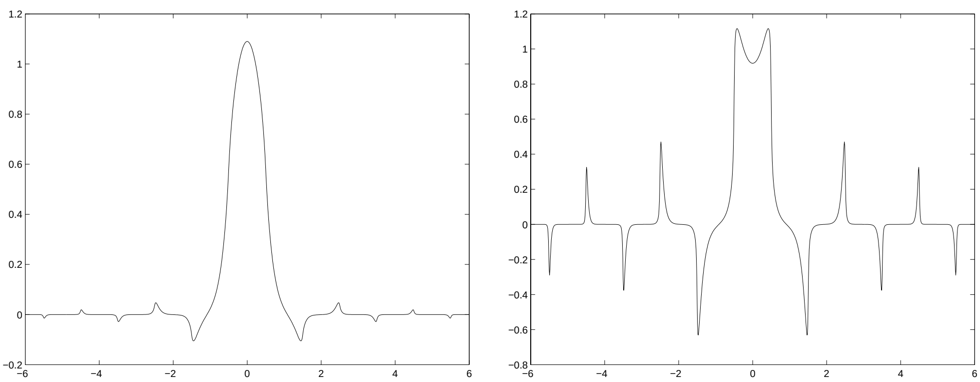


a b

Fig. 1. Plot of (a) $f^1(t)$ and (b) $f^{1/2}(t)$ for $-6 \le t \le 6$.



a b

Fig. 2. Plot of (a) $f^{1/4}(t)$ and (b) $f^{3/4}(t)$ for $-6 \le t \le 6$.