

On Lerch's transcendent and the Gaussian random walk

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Abstract

Let X_1, X_2, \dots be independent variables, each having a normal distribution with negative mean $-\beta < 0$ and variance 1. We consider the partial sums $S_n = X_1 + \dots + X_n$, with $S_0 = 0$, and refer to the process $\{S_n : n \geq 0\}$ as the Gaussian random walk. We present explicit expressions for the mean and variance of the maximum $M = \max\{S_n : n \geq 0\}$. These expressions are in terms of Taylor series about $\beta = 0$ with coefficients that involve the Riemann zeta function. Our results extend Kingman's first order approximation [19] of the mean for $\beta \downarrow 0$. We build upon the work of Chang & Peres [11], and use Bateman's formulas on Lerch's transcendent and Euler-Maclaurin summation as key ingredients.

1 Introduction

Let X_1, X_2, \dots be independent variables, each having a normal distribution with mean $-\beta < 0$ and variance 1. We consider the partial sums $S_n = X_1 + \dots + X_n$, with $S_0 = 0$, and refer to the process $\{S_n : n \geq 0\}$ as the Gaussian random walk. In this paper we present explicit expressions for several characteristics of the distribution of the maximum

$$M = \max\{S_n : n \geq 0\}. \tag{1.1}$$

The distribution of M plays an important role in several areas of applied probability. In queueing theory, it typically occurs in a regime called *heavy traffic* (see [2, 19, 17, 26]), in which the load is just below its critical level, and so the queue is only just stable with relatively large queue lengths and waiting times. For the limiting waiting time $W = \lim_{n \rightarrow \infty} W_n$, with $W_1 = 0$ and $W_{n+1} = (W_n + X_n)^+$, it follows from Spitzer's random-walk identities that W is, in distribution, equal to M . In the context of queues and heavy traffic, Kingman [19] was the first to remark the relevance of M in his 1965 paper. He noticed among other things:

Despite the apparent simplicity of the problem, there does not seem to be an explicit expression even for $\mathbb{E}M\dots$, but it is possible to give quite sharp inequalities and asymptotic results for small β .

Indeed, Kingman showed that, for $\beta \downarrow 0$,

$$\mathbb{E}M = \frac{1}{2\beta} - c + \mathcal{O}(\beta) \quad , \quad c \approx 0.58. \tag{1.2}$$

The tail distribution of M is tantamount to computing level crossing probabilities of the Gaussian random walk, i.e., for $x > 0$, $\{M > x\} = \{\tau(x) < \infty\}$, where $\tau(x) = \inf\{n \geq 1 : S_n > x\}$. This level-crossings interpretation makes that the tail distribution of M is important in *sequential analysis* and *risk theory*. Chang & Peres [11] derived an exact expression (2.1) for the expected value of the first ladder height $\mathbb{E}S_\tau$, with $\tau = \tau(0)$, which by the relation $\mathbb{E}S_\tau = \beta/\mathbb{P}(M = 0)$ leads to an exact expression for $\mathbb{P}(M = 0)$. They present $\mathbb{E}S_\tau$ as a Taylor series about $\beta = 0$ with coefficients that involve the Riemann zeta function, a considerable

achievement that generalizes first order approximations of Siegmund [22] and Chang [10].

Ladder heights fulfill an important role in probability theory, both in the exact analysis of random walks (see Asmussen [2], Feller [14]), and in the asymptotic analysis of boundary crossing problems (Siegmund [23]). In the latter case, a quantity of interest is the limiting expected overshoot, defined as $\mathbb{E}(S_\tau^2)/(2\mathbb{E}S_\tau)$ for $\beta = 0$. This quantity can be shown to be $-\zeta(1/2)/\sqrt{2\pi} = 0.5812\dots$, with $\zeta(z)$ the Riemann zeta function. The same quantity arises in sequentially testing for the drift of a Brownian motion [12], corrected diffusion approximations [22], simulation of Brownian motion [3, 9] and option pricing [7]. These applications have in common that a Brownian motion is observed only at equidistant sampling points. As it turns out¹, the c in (1.2) is in fact $-\zeta(1/2)/\sqrt{2\pi}$, so Kingman, albeit in disguised form, related $\mathbb{E}M$ to the Riemann zeta function already in 1965. We shall extend Kingman's approximation (1.2) to an explicit expression for $\mathbb{E}M$, in the same spirit as Chang & Peres extended the results of Siegmund [22] and Chang [10]. Moreover, we present a similar expression for the variance of M , to be denoted by $\text{Var}M$. The new expressions for $\mathbb{E}M$ and $\text{Var}M$ both concern Taylor series about zero with coefficients that involve the Riemann zeta function.

We first derive the Chang & Peres result (2.1) in our own fashion. Like Chang & Peres we start from a Spitzer-type expression for $\mathbb{P}(M = 0)$, take its derivative with respect to β , rewrite the derivative in terms of the Riemann zeta function, and finally integrate to obtain (2.1). For rewriting the derivative, Chang & Peres built upon the 1905 paper of Hardy [16] and present an analytic continuation of the function $\text{Li}_s(z) = \sum_{n=1}^{\infty} n^{-s}z^n$, known as the *polylogarithm* or *Jonquières function*. They were probably unaware of the fact that $\text{Li}_s(z)$ is a special case of *Lerch's transcendent*, see (2.4), for which the matter of analytic continuation has been established in full generality by Bateman (and/or the staff of the Bateman Manuscript Project), see [13], §1.11(8) and (2.5). Hence, although Chang & Peres [11] give a separate proof, their Theorem 2.1 should be attributed to Bateman.

Our derivation of (2.1)—that incorporates Bateman's formulas and an asymptotic determination of the integration constant—sets the stage for the derivation of the new explicit

¹Kingman [19] presents c as $(2\pi)^{-1/2} \sum_{n=1}^{\infty} [\sqrt{n}(\sqrt{n} + \sqrt{n-1})^2]^{-1/2}$, which by Euler-Maclaurin summation can be shown to be $-(2\pi)^{-1/2}\zeta(1/2)$. Similar relations are the topic of Problem 602 posed by Glasser & Boersma in [15].

expressions for $\mathbb{E}M$ and $\text{Var}M$. As an aside, we obtain the following asymptotic results for $\beta \downarrow 0$:

$$\mathbb{E}M = \frac{1}{2\beta} + \frac{\zeta(1/2)}{\sqrt{2\pi}} + \frac{1}{4}\beta + \mathcal{O}(\beta^2), \quad (1.3)$$

and

$$\text{Var}M = \frac{1}{4\beta^2} - \frac{1}{4} - \frac{2\zeta(-1/2)}{\sqrt{2\pi}}\beta - \frac{1}{24}\beta^2 + \mathcal{O}(\beta^3), \quad (1.4)$$

where $\zeta(1/2) \approx -1.4604$ and $\zeta(-1/2) \approx -0.2079$. In comparing (1.2) and (1.3), (1.3) contains an additional term $\frac{1}{4}\beta$. This term, and $-\frac{1}{24}\beta^2$ in (1.4), follow from a rather intricate application of the Euler-Maclaurin summation formula. The error terms in both (1.3) and (1.4) will be replaced by Taylor series with coefficients that involve the Riemann zeta function.

1.1 Structure of the paper

We present our main results in the next section. Sec. 3 is devoted to an exposition of our derivation of the Chang & Peres result. The proofs of the new expressions for the mean and variance of the maximum are given in Sec. 4 and Sec. 5, respectively. The new expressions for the mean and variance of M are alternatives for their Spitzer-type counterparts. The latter tend to converge more slowly for a decreasing drift β , whereas the opposite holds for the new expressions. We investigate this difference in speed of convergence in Sec. 6. Concluding remarks are made in Sec. 7.

2 Main results

We present three theorems. The first, on $\mathbb{P}(M = 0)$, is essentially due to Chang & Peres [11], but we give a separate proof in Sec. 3:

Theorem 2.1. (Chang & Peres [11]) *The probability that the maximum of the Gaussian random walk is zero satisfies*

$$\mathbb{P}(M = 0) = \sqrt{2}\beta \exp \left\{ \frac{\beta}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(1/2 - r)}{r!(2r + 1)} \left(\frac{-\beta^2}{2} \right)^r \right\}, \quad (2.1)$$

for $0 < \beta < 2\sqrt{\pi}$.

Then, largely motivated by Chang & Peres, but taking our own approach, we prove the next two theorems.

Theorem 2.2. *The expectation of the maximum of the Gaussian random walk satisfies*

$$\mathbb{E}M = \frac{1}{2\beta} + \frac{\zeta(1/2)}{\sqrt{2\pi}} + \frac{1}{4}\beta + \frac{\beta^2}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-1/2-r)}{r!(2r+1)(2r+2)} \left(\frac{-\beta^2}{2}\right)^r, \quad (2.2)$$

for $0 < \beta < 2\sqrt{\pi}$.

Theorem 2.3. *The variance of the maximum of the Gaussian random walk satisfies*

$$\text{Var}M = \frac{1}{4\beta^2} - \frac{1}{4} - \frac{2\zeta(-1/2)}{\sqrt{2\pi}}\beta - \frac{\beta^2}{24} - \frac{2\beta^3}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-3/2-r)}{r!(2r+1)(2r+2)(2r+3)} \left(\frac{-\beta^2}{2}\right)^r, \quad (2.3)$$

for $0 < \beta < 2\sqrt{\pi}$.

The key ingredients for obtaining the above series are Euler-Maclaurin summation and a result on Lerch's transcendent. Lerch's transcendent is defined as the analytic continuation of the series

$$\Phi(z, s, v) = \sum_{n=0}^{\infty} (v+n)^{-s} z^n, \quad (2.4)$$

which converges for any real number $v \neq 0, -1, -2, \dots$ if z and s are any complex numbers with either $|z| < 1$, or $|z| = 1$ and $\text{Re}(s) > 1$. Note that $\zeta(s) := \Phi(1, s, 1)$. We shall use the important result derived by Bateman [13], §1.11(8) (with $\zeta(s, v) := \Phi(1, s, v)$ the Hurwitz zeta function)

$$\Phi(z, s, v) = \frac{\Gamma(1-s)}{z^v} (\ln 1/z)^{s-1} + z^{-v} \sum_{r=0}^{\infty} \zeta(s-r, v) \frac{(\ln z)^r}{r!}, \quad (2.5)$$

which holds for $|\ln z| < 2\pi$, $s \neq 1, 2, 3, \dots$, and $v \neq 0, -1, -2, \dots$.

3 Proof of Theorem 2.1

From Spitzer's identity for random walks [24] we have

$$\mathbb{P}(M = 0) = \exp \left\{ - \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}(S_n > 0) \right\} = \exp \left\{ - \sum_{n=1}^{\infty} \frac{1}{n} P(-\beta\sqrt{n}) \right\}, \quad (3.1)$$

with $P(\cdot)$ the standard normal distribution function

$$P(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^a e^{-\frac{1}{2}x^2} dx. \quad (3.2)$$

The second equality in (3.1) follows from the normality of S_n .

With $\psi = \frac{1}{2}\beta^2$ and F defined by

$$F(\beta) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{\sqrt{2\pi}} \int_{\beta\sqrt{n}}^{\infty} e^{-\frac{1}{2}x^2} dx, \quad \beta > 0, \quad (3.3)$$

we have

$$\begin{aligned} F'(\beta) &= \frac{-1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{e^{-n\psi}}{\sqrt{n}} = \frac{-e^{-\psi}}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{e^{-n\psi}}{\sqrt{n+1}} \\ &= \frac{-e^{-\psi}}{\sqrt{2\pi}} \Phi(z = e^{-\psi}, s = \frac{1}{2}, v = 1). \end{aligned} \quad (3.4)$$

Then by (2.5), when $0 < \psi < 2\pi$,

$$\begin{aligned} F'(\beta) &= \frac{-e^{-\psi}}{\sqrt{2\pi}} \left[\frac{\Gamma(1/2)}{e^{-\psi}} \psi^{-1/2} + e^{\psi} \sum_{r=0}^{\infty} \zeta\left(\frac{1}{2} - r\right) \frac{(-\psi)^r}{r!} \right] \\ &= \frac{-1}{\sqrt{2}} \psi^{-1/2} - \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \zeta\left(\frac{1}{2} - r\right) \frac{(-\psi)^r}{r!}, \end{aligned} \quad (3.5)$$

with $\zeta(s)$ denoting the Riemann zeta function. Restoring β we get

$$F'(\beta) + \frac{1}{\beta} = \frac{-1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \zeta\left(\frac{1}{2} - r\right) \frac{(-\frac{1}{2}\beta^2)^r}{r!}, \quad 0 < \beta < 2\sqrt{\pi}. \quad (3.6)$$

The right-hand side of (3.6) is a well-behaved function of β , and integrating we get

$$F(\beta) + \ln \beta = L - \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(\frac{1}{2} - r)(-\frac{1}{2})^r \beta^{2r+1}}{r!(2r+1)}, \quad (3.7)$$

where $L = \lim_{\beta \downarrow 0} (F(\beta) + \ln \beta)$.

We shall show that $L = -\frac{1}{2} \ln 2$. To that end we note that

$$\begin{aligned} F(\beta) &= \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{\sqrt{\pi}} \int_{\sqrt{n\psi}}^{\infty} e^{-u^2} du \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{2}{\sqrt{\pi}} \int_{\sqrt{n\psi}}^{\infty} e^{-u^2} du - e^{-n\psi} \right) - \frac{1}{2} \ln(1 - e^{-\psi}) \\ &= \frac{1}{2} \psi \sum_{n=1}^{\infty} \frac{1}{n\psi} \left(\frac{2}{\sqrt{\pi}} \int_{\sqrt{n\psi}}^{\infty} e^{-u^2} du - e^{-n\psi} \right) - \ln \beta + \frac{1}{2} \ln 2 + o(1) \end{aligned} \quad (3.8)$$

as $\beta = (2\psi)^{1/2} \downarrow 0$. The function

$$g(y) := \frac{1}{y} \left(\frac{2}{\sqrt{\pi}} \int_{\sqrt{y}}^{\infty} e^{-u^2} du - e^{-y} \right), \quad y > 0, \quad (3.9)$$

decays exponentially as $y \rightarrow \infty$ while $g(y) = \mathcal{O}(y^{-1/2})$, $y \downarrow 0$. It is then routine to show that

$$\frac{1}{2} \psi \sum_{n=0}^{\infty} g(n\psi) \rightarrow \frac{1}{2} \int_0^{\infty} g(y) dy, \quad \psi \downarrow 0. \quad (3.10)$$

The latter integral can be evaluated as

$$\begin{aligned} &\int_0^{\infty} \frac{1}{y} \left(\frac{2}{\sqrt{\pi}} \int_{\sqrt{y}}^{\infty} e^{-u^2} du - e^{-y} \right) dy = \\ &\left(\frac{2}{\sqrt{\pi}} \int_{\sqrt{y}}^{\infty} e^{-u^2} du - e^{-y} \right) \ln y \Big|_0^{\infty} - \int_0^{\infty} \left(\frac{2}{\sqrt{\pi}} \cdot -\frac{1}{2} y^{-1/2} \cdot e^{-y} + e^{-y} \right) \ln y dy = \\ &\frac{1}{\sqrt{\pi}} \int_0^{\infty} y^{-1/2} e^{-y} \ln y dy - \int_0^{\infty} e^{-y} \ln y dy = \frac{1}{\sqrt{\pi}} \Gamma'(1/2) - \Gamma'(1) = -2 \ln 2, \end{aligned} \quad (3.11)$$

by Abramowitz-Stegun [1] §6.3.1-4 on p. 258. Hence, $L = -\frac{1}{2} \ln 2$ indeed, and so it is shown

that, for $0 < \beta < 2\sqrt{\pi}$, we have

$$F(\beta) = -\ln \beta - \frac{1}{2} \ln 2 - \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(\frac{1}{2} - r)(-\frac{1}{2})^r \beta^{2r+1}}{r!(2r+1)}, \quad (3.12)$$

which, by (3.1) completes the proof of Thm. 2.1.

To recapitulate, we started from the Spitzer-type expression (3.1), rewrote its derivative (3.3) in terms of Lerch's transcendent (3.4), applied Bateman's formulas to obtain a Taylor series (3.5), integrated the Taylor series (3.7), and finally determined the integration constant L .

Remark 3.1. The integration constant could have been determined from the relation $\mathbb{P}(M = 0) = \beta/\mathbb{E}S_\tau$ (with $\mathbb{E}S_\tau$ the expected value of the first ladder height, see Sec. 1), and using the fact that $\mathbb{E}S_\tau = 1/\sqrt{2}$ for $\beta = 0$, as proven by Spitzer [25]; see also Lai [20]. Alternatively, one could use the first order approximation in Jelenkovic et al. [17], that is, $\mathbb{P}(M = 0) = \sqrt{2}\beta(1 + o(1))$ as $\beta \downarrow 0$. The primary purpose of this section, however, is to set the stage for the next two sections, in which there is no other way of determining integration constants than to apply asymptotic methods.

4 Proof of Theorem 2.2

From Spitzer's identity [24] we know that

$$\mathbb{E}M = \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}(S_n^+) = \sum_{n=1}^{\infty} \left(\frac{e^{-\frac{1}{2}n\beta^2}}{\sqrt{2\pi n}} - \beta P(-\beta\sqrt{n}) \right). \quad (4.1)$$

With $\psi = \frac{1}{2}\beta^2$ we have

$$\sum_{n=1}^{\infty} \frac{e^{-\frac{1}{2}n\beta^2}}{\sqrt{2\pi n}} = \frac{e^{-\psi}}{\sqrt{2\pi}} \Phi(z = e^{-\psi}, s = \frac{1}{2}, v = 1) = \frac{1}{\beta} + \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(\frac{1}{2} - r)}{r!} (-\psi)^r. \quad (4.2)$$

Now we consider

$$G(\beta) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\beta\sqrt{n}}^{\infty} e^{-x^2/2} dx. \quad (4.3)$$

We have

$$\begin{aligned}
G'(\beta) &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \sqrt{n} \cdot -e^{-n\psi} \\
&= \frac{-e^{-\psi}}{\sqrt{2\pi}} \Phi(z = e^{-\psi}, s = -\frac{1}{2}, v = 1).
\end{aligned} \tag{4.4}$$

Then by (2.5), when $\psi < 2\pi$,

$$\begin{aligned}
G'(\beta) &= \frac{-e^{-\psi}}{\sqrt{2\pi}} \left[\frac{\Gamma(3/2)}{e^{-\psi}} \psi^{-3/2} + e^{\psi} \sum_{r=0}^{\infty} \zeta(-\frac{1}{2} - r) \frac{(-\psi)^r}{r!} \right] \\
&= \frac{-1}{2\sqrt{2}} \psi^{-3/2} - \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \zeta(-\frac{1}{2} - r) \frac{(-\psi)^r}{r!}.
\end{aligned} \tag{4.5}$$

Therefore, restoring β , we get

$$G'(\beta) = -\beta^{-3} - H(\beta) \quad ; \quad H(\beta) = \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-\frac{1}{2} - r) (-1/2)^r}{r!} \beta^{2r}. \tag{4.6}$$

We note that $H(\beta)$ is well-behaved in $0 \leq \beta < 2\sqrt{\pi}$, and that

$$\frac{d}{d\beta} \left[G(\beta) - \frac{1}{2\beta^2} \right] = G'(\beta) + \frac{1}{\beta^3} = -H(\beta). \tag{4.7}$$

By integration from 0 to β we thus get

$$\begin{aligned}
G(\beta) - \frac{1}{2\beta^2} - \lim_{\varepsilon \downarrow 0} \left(G(\varepsilon) - \frac{1}{2\varepsilon^2} \right) &= - \int_0^{\beta} H(\beta_1) d\beta_1 \\
&= \frac{-1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-\frac{1}{2} - r) (-1/2)^r}{r!(2r+1)} \beta^{2r+1}.
\end{aligned} \tag{4.8}$$

We shall show that

$$\lim_{\varepsilon \downarrow 0} \left(G(\varepsilon) - \frac{1}{2\varepsilon^2} \right) = -\frac{1}{4}. \tag{4.9}$$

To that end we use the Euler-Maclaurin summation formula (see De Bruijn [8], Sec. 3.6,

pp. 40-42)

$$\begin{aligned} \sum_{n=1}^N f(n) &= \int_1^N f(x)dx + \frac{1}{2}f(1) + \frac{1}{2}f(N) \\ &+ \sum_{k=1}^m \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(N) - f^{(2k-1)}(1) \right) - \int_1^N f^{(2m)}(x) \frac{B_{2m}(x - \lfloor x \rfloor)}{(2m)!} dx, \end{aligned} \quad (4.10)$$

with $m = 1$, $N \rightarrow \infty$ and

$$f_\delta(x) = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\delta x}}^{\infty} e^{-u^2} du =: g(\delta x) \quad ; \quad \delta = \frac{1}{2}\varepsilon^2. \quad (4.11)$$

Hence

$$\begin{aligned} G_N(\varepsilon) &= \sum_{n=1}^N \frac{1}{\sqrt{2\pi}} \int_{\varepsilon\sqrt{n}}^{\infty} e^{-x^2/2} dx = \sum_{n=1}^N f_\delta(n) \\ &= \int_1^N g(\delta x) dx + \frac{1}{2}g(\delta) + \frac{1}{2}g(N\delta) \\ &+ \frac{1}{2}B_2 (g'(N\delta) - g'(\delta)) \delta - \int_1^N \delta^2 g''(\delta x) \frac{B_2(x - \lfloor x \rfloor)}{2} dx. \end{aligned} \quad (4.12)$$

Letting $N \rightarrow \infty$ and noting that for $g(y) = \frac{1}{\sqrt{\pi}} \int_{\sqrt{y}}^{\infty} e^{-u^2} du$ there holds that $g, g', g'' \rightarrow 0$ exponentially fast as $y \rightarrow \infty$, we get

$$G(\varepsilon) = \int_1^{\infty} g(\delta x) dx + \frac{1}{2}g(\delta) - \frac{1}{2}B_2 g'(\delta) \delta - \frac{1}{2} \int_1^{\infty} \delta^2 g''(\delta x) \frac{B_2(x - \lfloor x \rfloor)}{2} dx. \quad (4.13)$$

The last integral at the right-hand side of (5.11) can be bounded by

$$\int_1^{\infty} \delta^2 |g''(\delta x)| \frac{1}{2} B_2 dx = \frac{1}{12} \delta \int_\delta^{\infty} |g''(y)| dy. \quad (4.14)$$

We further get

$$g'(y) = -\frac{e^{-y}}{2\sqrt{\pi y}} \quad , \quad g''(y) = \frac{e^{-y}}{4y\sqrt{\pi y}}(2y+1) \geq 0. \quad (4.15)$$

Therefore, we see that

$$\delta g'(\delta) = \mathcal{O}(\delta^{1/2}) \quad , \quad \delta \int_{\delta}^{\infty} |g''(y)| dy = -\delta g'(\delta) = \mathcal{O}(\delta^{1/2}). \quad (4.16)$$

Furthermore,

$$\begin{aligned} \int_1^{\infty} g(\delta x) dx &= \delta^{-1} \int_{\delta}^{\infty} g(y) dy = \delta^{-1} \int_{\delta}^{\infty} \left(\frac{1}{\sqrt{\pi}} \int_{\sqrt{y}}^{\infty} e^{-u^2} du \right) dy \\ &= \delta^{-1} \int_0^{\infty} \left(\frac{1}{\sqrt{\pi}} \int_{\sqrt{y}}^{\infty} e^{-u^2} du \right) dy - \delta^{-1} \int_0^{\delta} \left(\frac{1}{\sqrt{\pi}} \int_{\sqrt{y}}^{\infty} e^{-u^2} du \right) dy. \end{aligned} \quad (4.17)$$

Then from $g(\delta) = \frac{1}{2} + \mathcal{O}(\delta^{1/2})$ we get

$$\delta^{-1} \int_0^{\delta} \left(\frac{1}{\sqrt{\pi}} \int_{\sqrt{y}}^{\infty} e^{-u^2} du \right) dy = \frac{1}{2} + \mathcal{O}(\delta^{1/2}), \quad (4.18)$$

and

$$\begin{aligned} \int_0^{\infty} \left(\frac{1}{\sqrt{\pi}} \int_{\sqrt{y}}^{\infty} e^{-u^2} du \right) dy &= \frac{1}{\sqrt{\pi}} y \int_{\sqrt{y}}^{\infty} e^{-u^2} du \Big|_0^{\infty} - \int_0^{\infty} y \frac{1}{\sqrt{\pi}} \frac{1}{2} y^{-1/2} \cdot -e^{-y} dy \\ &= \frac{1}{2\sqrt{\pi}} \int_0^{\infty} y^{1/2} e^{-y} dy = \frac{1}{4}. \end{aligned} \quad (4.19)$$

Therefore

$$\int_1^{\infty} g(\delta x) dx = \frac{1}{4\delta} - \frac{1}{2} + \mathcal{O}(\delta^{1/2}) \quad , \quad \delta \downarrow 0. \quad (4.20)$$

It finally follows that

$$G(\varepsilon) = \left(\frac{1}{4\delta} - \frac{1}{2} + \mathcal{O}(\delta^{1/2}) \right) + \frac{1}{2} \left(\frac{1}{2} + \mathcal{O}(\delta^{1/2}) \right) \quad ; \quad \delta = \frac{1}{2}\varepsilon^2, \quad (4.21)$$

and we obtain (4.9). It is thus concluded that

$$G(\beta) = \frac{1}{2\beta^2} - \frac{1}{4} - \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-\frac{1}{2}-r)(-1/2)^r}{r!(2r+1)} \beta^{2r+1}. \quad (4.22)$$

Combining (4.1), (4.2) and (4.22), we then obtain

$$\begin{aligned}
\mathbb{E}M &= \sum_{n=1}^{\infty} \frac{e^{-\frac{1}{2}n\beta^2}}{\sqrt{2\pi n}} - \beta \sum_{n=1}^{\infty} P(-\beta\sqrt{n}) = \sum_{n=1}^{\infty} \frac{e^{-\frac{1}{2}n\beta^2}}{\sqrt{2\pi n}} - \beta G(\beta) \\
&= \frac{1}{\beta} + \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(\frac{1}{2}-r)(-1/2)^r}{r!} \beta^{2r} - \beta \left[\frac{1}{2\beta^2} - \frac{1}{4} - \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-\frac{1}{2}-r)(-1/2)^r}{r!(2r+1)} \beta^{2r+1} \right] \\
&= \frac{1}{2\beta} + \frac{1}{4}\beta + \frac{1}{\sqrt{2\pi}} \left\{ \sum_{r=0}^{\infty} \frac{\zeta(\frac{1}{2}-r)(-1/2)^r}{r!} \beta^{2r} + \sum_{r=0}^{\infty} \frac{\zeta(-\frac{1}{2}-r)(-1/2)^r}{r!(2r+1)} \beta^{2r+2} \right\}.
\end{aligned} \tag{4.23}$$

Splitting off the term with $r = 0$ and replacing the summation index $r = 1, 2, \dots$ by $r + 1$, $r = 0, 1, \dots$ in the first series in (4.23), we get

$$\mathbb{E}M = \frac{1}{2\beta} + \frac{1}{4}\beta + \frac{1}{\sqrt{2\pi}} \zeta\left(\frac{1}{2}\right) + \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-\frac{1}{2}-r)(-1/2)^r \beta^{2r+2}}{r!(2r+1)(2r+2)}. \tag{4.24}$$

5 Proof of Theorem 2.3

From Spitzer's identity [24] we get

$$\text{Var}M = \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}((S_n^+)^2), \tag{5.1}$$

which, using the normality of S_n , yields

$$\begin{aligned}
\text{Var}M &= \sum_{n=1}^{\infty} \frac{1}{n\sqrt{2\pi}} \int_{\beta\sqrt{n}}^{\infty} (x\sqrt{n} - \beta n)^2 e^{-x^2/2} dx \\
&= \sum_{n=1}^{\infty} \left((\beta^2 n + 1) P(-\beta\sqrt{n}) - \frac{\beta}{\sqrt{2\pi}} \sqrt{ne}^{-\beta^2 n/2} \right),
\end{aligned} \tag{5.2}$$

where the second equality in (5.2) follows from partial integration. We have established earlier, see (4.6) that

$$\frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \sqrt{ne}^{-\beta^2 n/2} = \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-\frac{1}{2}-r)(-1/2)^r}{r!} \beta^{2r} + \frac{1}{\beta^3}. \tag{5.3}$$

Therefore, it remains to evaluate

$$I(\beta) = \sum_{n=1}^{\infty} nP(-\beta\sqrt{n}) = \sum_{n=1}^{\infty} \frac{n}{\sqrt{2\pi}} \int_{\beta\sqrt{n}}^{\infty} e^{-x^2/2} dx, \quad (5.4)$$

and to combine the results with (5.3) and (4.22) according to (5.2).

There holds, with $\psi = \frac{1}{2}\beta^2$ as earlier,

$$I'(\beta) = -\sum_{n=1}^{\infty} \frac{n^{3/2}}{\sqrt{2\pi}} e^{-n\psi} = \frac{-e^{-\psi}}{\sqrt{2\pi}} \Phi(z = e^{-\psi}, s = -\frac{3}{2}, v = 1), \quad (5.5)$$

and by Bateman's result (2.5),

$$\begin{aligned} I'(\beta) &= \frac{-e^{-\psi}}{\sqrt{2\pi}} \left[\frac{\Gamma(5/2)}{e^{-\psi}} \psi^{-5/2} + e^{\psi} \sum_{r=0}^{\infty} \zeta(-\frac{3}{2} - r) \frac{(-\psi)^r}{r!} \right] \\ &= \frac{-3}{4\sqrt{2}} \psi^{-5/2} - \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \zeta(-\frac{3}{2} - r) \frac{(-\psi)^r}{r!} \\ &= -3\beta^{-5} - \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-\frac{3}{2} - r)(-1/2)^r}{r!} \beta^{2r}, \end{aligned} \quad (5.6)$$

assumed that $0 < \beta < 2\sqrt{\pi}$. The series on the last line of (5.6) is well-behaved in $0 \leq \beta < 2\sqrt{\pi}$, whence $I'(\beta) + 3\beta^{-5}$ is integrable, and we obtain

$$I(\beta) - \frac{3}{4}\beta^{-4} = \lim_{\varepsilon \downarrow 0} (I(\varepsilon) - \frac{3}{4}\varepsilon^{-4}) - \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-\frac{3}{2} - r)(-1/2)^r}{r!(2r+1)} \beta^{2r+1}. \quad (5.7)$$

We shall show that

$$\lim_{\varepsilon \downarrow 0} (I(\varepsilon) - \frac{3}{4}\varepsilon^{-4}) = -\frac{1}{24} \quad (5.8)$$

by applying the Euler-Maclaurin summation formula (4.10) with $m = 1$, $N \rightarrow \infty$ as before.

We consider now

$$f_{\delta}(x) = \frac{\delta x}{\sqrt{\pi}} \int_{\sqrt{\delta x}}^{\infty} e^{-u^2} du =: h(\delta x) \quad ; \quad \delta = \frac{1}{2}\varepsilon^2, \quad (5.9)$$

in which

$$h(x) = xg(x) \quad ; \quad g(x) = \frac{1}{\sqrt{\pi}} \int_{\sqrt{x}}^{\infty} e^{-u^2} du, \quad x \geq 0. \quad (5.10)$$

Then

$$I(\varepsilon) = \frac{1}{\delta} \left[\int_1^\infty h(\delta x) dx + \frac{1}{2}h(\delta) - \frac{1}{2}B_2h'(\delta)\delta - \int_1^\infty \delta^2 h''(\delta x) \frac{B_2(x - \lfloor x \rfloor)}{2} dx \right]. \quad (5.11)$$

Next we shall take $\delta \downarrow 0$, and to that end we see that

$$\frac{1}{\delta}h(\delta) = g(\delta) \rightarrow \frac{1}{2} \quad ; \quad h'(\delta) = g(\delta) - \frac{\delta^{1/2}}{2\sqrt{\pi}}e^{-\delta} \rightarrow \frac{1}{2} \quad , \quad \delta \downarrow 0. \quad (5.12)$$

Furthermore,

$$\frac{1}{\delta} \int_1^\infty h(\delta x) dx = \frac{1}{\delta^2} \int_0^\infty h(x) dx - \frac{1}{\delta^2} \int_0^\delta h(x) dx, \quad (5.13)$$

in which

$$\frac{1}{\delta^2} \int_0^\delta xg(x) dx \rightarrow \frac{1}{4} + \mathcal{O}(\delta^{1/2}). \quad (5.14)$$

Also, by partial integration,

$$\begin{aligned} \int_0^\infty h(x) dx &= \int_0^\infty x \frac{1}{\sqrt{\pi}} \left(\int_{\sqrt{x}}^\infty e^{-u^2} du \right) dx \\ &= \frac{x^2}{2\sqrt{\pi}} \int_{\sqrt{x}}^\infty e^{-u^2} du \Big|_0^\infty - \int_0^\infty \frac{x^2}{2\sqrt{\pi}} \cdot -\frac{1}{2}x^{-1/2}e^{-x} dx \\ &= \frac{1}{4\sqrt{\pi}} \int_0^\infty x^{3/2}e^{-x} dx = \frac{1}{4\sqrt{\pi}}\Gamma(5/2) = \frac{3}{16}. \end{aligned} \quad (5.15)$$

Therefore

$$\frac{1}{\delta} \int_1^\infty h(\delta x) dx = \frac{3}{16\delta^2} - \frac{1}{4} + \mathcal{O}(\delta^{1/2}). \quad (5.16)$$

Finally,

$$\begin{aligned} h''(x) &= (xg(x))'' = 2g'(x) + xg''(x) \\ &= \frac{1}{2\sqrt{\pi x}}(x - \frac{3}{2})e^{-x} \in L^1([0, \infty)), \end{aligned} \quad (5.17)$$

and

$$\frac{1}{2}B_2(x - \lfloor x \rfloor) = \frac{1}{2\pi^2} \sum_{k=1}^\infty \frac{\cos 2\pi kx}{k^2}, \quad (5.18)$$

see De Bruijn [8], p. 41. Therefore,

$$\begin{aligned}
\delta \int_1^\infty h''(\delta x) \frac{1}{2} B_2(x - \lfloor x \rfloor) dx &= \int_\delta^\infty h''(x) \frac{1}{2} B_2(x/\delta - \lfloor x/\delta \rfloor) dx \\
&= \frac{1}{2\pi^2} \sum_{k=1}^\infty \frac{1}{k^2} \int_\delta^\infty h''(x) \cos(2\pi kx/\delta) dx \rightarrow 0 \quad , \quad \delta \downarrow 0,
\end{aligned} \tag{5.19}$$

since $\int_\delta^\infty h''(x) \cos(2\pi kx/\delta) dx \rightarrow 0$ as $\delta \downarrow 0$ by the Riemann-Lebesgue lemma on Fourier integrals. Putting this altogether, we find (recall $\delta = \frac{1}{2}\varepsilon^2$)

$$\lim_{\varepsilon \downarrow 0} (I(\varepsilon) - \frac{3}{4}\varepsilon^{-4}) = -\frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{2} - 0 = -\frac{1}{24}. \tag{5.20}$$

Hence we obtain for $0 < \beta < 2\sqrt{\pi}$

$$I(\beta) = \frac{3}{4}\beta^{-4} - \frac{1}{24} - \frac{1}{\sqrt{2\pi}} \sum_{r=0}^\infty \frac{\zeta(-\frac{3}{2} - r)(-1/2)^r}{r!(2r+1)} \beta^{2r+1}. \tag{5.21}$$

We insert this result, together with (5.3) and (4.22), into (5.2) and get

$$\begin{aligned}
\text{Var}W &= -\frac{1}{\sqrt{2\pi}} \sum_{r=0}^\infty \frac{\zeta(-\frac{3}{2} - r)(-1/2)^r}{r!(2r+1)} \beta^{2r+3} + \frac{3}{4}\beta^{-2} - \frac{\beta^2}{24} \\
&\quad - \frac{1}{\sqrt{2\pi}} \sum_{r=0}^\infty \frac{\zeta(-\frac{1}{2} - r)(-1/2)^r}{r!(2r+1)} \beta^{2r+1} + \frac{1}{2}\beta^{-2} - \frac{1}{4} \\
&\quad - \frac{1}{\sqrt{2\pi}} \sum_{r=0}^\infty \frac{\zeta(-\frac{1}{2} - r)(-1/2)^r}{r!} \beta^{2r+1} - \beta^{-2}.
\end{aligned} \tag{5.22}$$

Splitting off the terms with $r = 0$ and replacing the summation index $r = 1, 2, \dots$ by $r + 1$,

$r = 0, 1, \dots$ in the last two series in the right-hand side of (5.22), we get

$$\begin{aligned}
\text{Var}W &= \frac{1}{4\beta^2} - \frac{1}{4} - \frac{2\zeta(-1/2)}{\sqrt{2\pi}}\beta - \frac{1}{24}\beta^2 - \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \left\{ \frac{\zeta(-\frac{3}{2}-r)(-1/2)^r}{r!(2r+1)} \beta^{2r+3} \right. \\
&\quad \left. + \frac{\zeta(-\frac{3}{2}-r)(-1/2)^{r+1}}{(r+1)!(2r+3)} \beta^{2r+3} + \frac{\zeta(-\frac{3}{2}-r)(-1/2)^{r+1}}{(r+1)!} \beta^{2r+3} \right\} \\
&= \frac{1}{4\beta^2} - \frac{1}{4} - \frac{2\zeta(-1/2)}{\sqrt{2\pi}}\beta - \frac{1}{24}\beta^2 - \frac{2}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-\frac{3}{2}-r)(-1/2)^r \beta^{2r+3}}{r!(2r+1)(2r+2)(2r+3)}.
\end{aligned} \tag{5.23}$$

6 Convergence comparison Spitzer formulas and Lerch series

It is immediately clear that the infinite series in (2.1), (2.2) and (2.3) converge more rapidly for smaller values of β , while the contrary holds for their Spitzer-type counterparts (3.1), (4.1) and (5.1). To exemplify this difference in speed of convergence, we consider (4.2), i.e.,

$$\sum_{n=1}^{\infty} \frac{e^{-\frac{1}{2}n\beta^2}}{\sqrt{2\pi n}} = \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(\frac{1}{2}-r)}{r!} \left(-\frac{\beta^2}{2}\right)^r. \tag{6.1}$$

The left-hand side series converges for all $\beta > 0$ while the right-hand side series converges for all $\beta \in \mathbb{C}$, $|\beta| < 2\sqrt{\pi}$. From Whittaker & Watson [27] §13.151 (p. 269),

$$2^{1-s}\Gamma(s)\zeta(s)\cos(\frac{1}{2}s\pi) = \pi^s\zeta(1-s). \tag{6.2}$$

With $s = r + \frac{1}{2}$, the asymptotics of the Γ -function and the fact that $\zeta(r + 1/2) \rightarrow 1$ as $r \rightarrow \infty$, we see that

$$\left| \frac{1}{\sqrt{2\pi}} \frac{\zeta(\frac{1}{2}-r)}{r!} \left(-\frac{\beta^2}{2}\right)^r \right| \approx \frac{1}{\pi\sqrt{2r+1}} \left(\frac{\beta^2}{4\pi}\right)^r, \quad r \rightarrow \infty. \tag{6.3}$$

Hence, for comparing the convergence rates of the two series in (6.1), it is enough to find the point $\beta_0 > 0$ such that

$$e^{-\frac{1}{2}\beta_0^2} = \frac{\beta_0^2}{4\pi}. \tag{6.4}$$

With $x = \frac{1}{2}\beta^2$ we need to solve $x_0 e^{x_0} = 2\pi$ with $x_0 > 0$. This yields $x_0 = 1.4597$, $\beta_0 = 1.7086$, and the common value of the two members in (6.4) equals 0.2323. See [6], Sec. 2, where a

similar strategy is developed in connection with the evaluation of Legendre's chi-function.

7 Concluding remarks

Throughout this paper, the increments X_i were assumed to have variance 1. Results for the more general case in which the X_i are normally distributed with mean $\hat{\beta}\sigma$ and variance σ^2 readily follow from

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}((S_n^+)^k) = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{2\pi}} \int_{\hat{\beta}\sqrt{n}}^{\infty} \sigma^k (\sqrt{n}x - \hat{\beta}n)^k e^{-x^2/2} dx, \quad k = 1, 2, \dots \quad (7.1)$$

Results for higher moments of M may be obtained along the same lines as the mean and variance, starting from (7.1).

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