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Connecting renewal age processes and M/D/1 processor sharing queues through stick breaking

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Abstract: The renewal age process increases linearly with slope one and is reset to zero at points governed by a Poisson process. We present various results for the random variable H_x that represents the first time the process hits the level x. These results include three characterizations of the distribution function and asymptotic expressions for the tail distribution. The latter involve complex-valued solutions of the Lambert W function. We further establish several connections to other probabilistic models. Using the theory of uniform spacings, we show that H_x has the same distribution as the sojourn time of the first customer in an M/D/1 processor sharing queue.

Contents

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1 Outline

The renewal age process is a piecewise deterministic Markov process that increases linearly with slope one and is reset to zero at points governed by a Poisson process with rate λ. The principal subject of this paper is the random variable H_x that represents the first time the renewal age process hits level x, and that satisfies the stochastic equation

$$H_x \stackrel{d}{=} \mathbb{1}_{\{Z\geq x\}}x + \mathbb{1}_{\{Z<x\}}(H_x + Z), \tag{1}$$

where $\stackrel{d}{=}$ denotes equality in distribution, and Z is an exponentially distributed random variable with mean $1/\lambda$, independent of H_x.

We show in Section 2 that H_x can be represented as a geometric sum of random variables, from which its Laplace-Stieltjes transform immediately follows. We also prove that H_x has the same distribution as the sojourn time of the first customer in an M/D/1 processor sharing queue (with arrival rate λ and deterministic service time x). To establish this equivalence, we strongly rely on results for the order statistics of uniform spacings, also known as stick breaking (see [7, 13]). We further show that the stationary waiting time in the M/D/1 first-come-first-served queue can be fully expressed in terms of H_x. We hence establish a connection between each of these models through the first hitting time of the renewal age process.

Further properties of H_x are reported in Section 3. For H_x, and two related random variables, we provide stochastic recursive equations and scaling limits. Also, in Subsections 3.3-3.5, we provide three alternative expressions for the distribution function of H_x. The first expression follows from the connection to stick breaking and uses Whitworth's formula for the maximal uniform spacing (see for instance [7, 13]). The second expression follows from the observation that the distribution function is the solution to a certain differential-difference equation as studied in [2]. The third expression is obtained by using the Laplace inversion formula and involves the infinitely many singularities of the Laplace transform of H_x. These singularities are in fact expressible in terms of the Lambert W function (see [6]). The leading term, corresponding to the dominant singularity, leads to a sharp asymptotic expression for the tail of the distribution of H_x, which is shown to be different for the cases $\lambda x < 1$, $\lambda x = 1$ and $\lambda x > 1$. The second and higher terms of the expression involve the non-principal branches of the Lambert W function, for which several results are presented in Section 4.

For a random variable X, we denote the mean by $\mathbb{E}X$, the Laplace-Stieltjes transform by $\Phi_X(s) = \mathbb{E}(e^{-sX})$, and the distribution function by $F_X(t) = \mathbb{P}(X \leq t)$ and $\overline{F}_X(t) = \mathbb{P}(X > t)$.

2 Connecting two processes

In Subsection 2.1 we give a formal description of the renewal age process, as well as a characterization of the first hitting time H_x in terms of a geometric sum and the Laplace-Stieltjes transform. In Subsection 2.2 we first present several known results on uniform spacings and then show that the sojourn time of the first customer in an M/D/1 processor sharing queue is in distribution equal to H_x.

2.1 Renewal age process

Consider a renewal process with renewal epochs $t_1, t_2, \ldots$ governed by a Poisson process with rate λ, and let $\tau_n = \sum_{k=1}^{n} t_k$ denote their partial sum. The associated renewal age process $(A_t)_{t\geq 0}$ measures the time since the last epoch, so $A_t = t - \tau_{N_t}$ where $N_t = \sup\{n \in \mathbb{N} | \tau_n \leq t\}$.

For some $x \in \mathbb{R}$ let t_{K_x} be the first epoch with length greater or equal to x, i.e., $K_x = \inf\{n \in \mathbb{N} | t_n \geq x\}$. It immediately follows that K_x has a geometric distribution

$$\mathbb{P}(K_x = k) = e^{-\lambda x}(1 - e^{-\lambda x})^{k-1} \quad , \quad k = 1, 2, \ldots . \tag{2}$$

Let $T_x = \tau_{K_x - 1}$ and $R_x = \tau_{K_x}$ denote the beginning and the end of the epoch t_{K_x}; see Figure 1. Note that T_x can be described as the geometric sum

$$T_x \stackrel{d}{=} \sum_{i=1}^{K_x - 1} B_i, \tag{3}$$

where K_x is geometrically distributed as in (2), and $B, B_1, B_2, \ldots$ are i.i.d. exponential random variables with rate λ truncated at x. Note that the distribution of B depends on the parameter $e^{-\lambda x}$ of the geometric distribution.

Due to the lack-of-memory property of the exponential distribution the random variable $Z = t_{K_x} - x$ has again an exponential distribution and is independent of T_x. Let H_x denote the first time that the age process $(A_t)_{t\geq 0}$ hits the level x. Since $R_x = T_x + t_{K_x}$ it follows that

$$T_x \leq T_x + x = H_x \leq H_x + Z \stackrel{d}{=} R_x. \tag{4}$$

Theorem 1.

$$\Phi_{T_x}(s) = \frac{\lambda + s}{\lambda e^{-sx} + s e^{\lambda x}}. \tag{5}$$

Proof. The Laplace-Stieltjes transform of B is

$$\Phi_B(s) = \frac{1}{1 - e^{-\lambda x}} \int_0^x e^{-su} \lambda e^{-\lambda u} \, du = \frac{\lambda}{\lambda + s} \frac{1 - e^{-(\lambda + s)x}}{1 - e^{-\lambda x}}.$$

From (2) we get $\Phi_{T_x}(s) = \sum_{k=1}^{\infty} e^{-\lambda x}(1 - e^{-\lambda x})^{k-1}(\Phi_B(s))^{k-1}$, which leads to (5). □

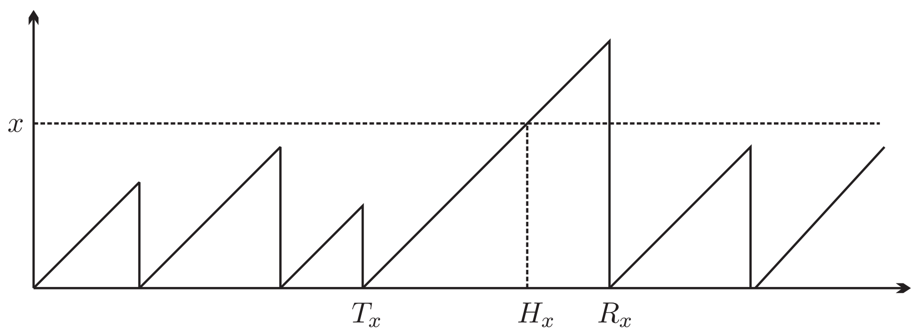

Figure 1: The renewal age process $(A_t)_{t\geq 0}$, with hitting time H_x and associated random variables T_x and R_x.

It follows from (4) and (5) that

$$\Phi_{R_x}(s) = \frac{\lambda}{\lambda + s e^{(\lambda+s)x}}, \quad \Phi_{H_x}(s) = \frac{\lambda + s}{\lambda + s e^{(\lambda+s)x}}. \tag{6}$$

Remark 2. There is another way to derive $\Phi_{H_x}(s)$. The Markov process $(A_t)_{t\geq 0}$ has an infinitesimal generator given by

$$\mathscr{A} f(x) = f'(x) - \lambda f(x) + \lambda f(0).$$

It is known that $f(X_t)\exp(-\int_0^t \mathscr{A} f(X_s)/f(X_s)\ ds)$ is a martingale (see [11]) and it is easy to show that the function $f_s(x) = 1/\Phi_{H_x}(s)$ fulfills $\mathscr{A} f_s(x) = s f_s(x)$. Consequently, $f_s(X_t)e^{-st}$ is a martingale. Optional stopping at H_x then leads to $\mathbb{E}(e^{-sH_x}) = 1/f_s(x) = \Phi_{H_x}(s)$ as required. Note that by using this approach we can derive

$$\mathbb{E}_a(e^{-sH_x}) = f_s(a)/f_s(x) = \frac{\lambda + s e^{(\lambda+s)a}}{\lambda + s e^{(\lambda+s)x}} \ , \quad a < x,$$

where $\mathbb{E}_a$ denotes the conditional expectation if we start the process A_t at a.

2.2 M/D/1 processor sharing queue

In this section we establish a connection between the renewal age process and the M/D/1 processor sharing queue. In order to do so, we first state some known results for uniform spacings.

Consider a stick of length t that is broken into n pieces, where the breaking points are given by a sample of size $n-1$ from the uniform distribution on $[0, t]$. Denote the breaking points by $U_1 < U_2 < \dots U_{n-1}$ and define the length of the pieces by $S_1 = U_1, S_2 = U_2 - U_1, \dots, S_n = t - U_{n-1}$. Throughout we let $Z, Z_1, Z_2, \dots$ denote independent and exponentially distributed random variables with mean $1/\lambda$ and N_t a Poisson process with rate λ. Let $Z_{(1)}, \dots, Z_{(n)}$ denote the order statistics such that $Z_{(k)}$ is the kth smallest value among $Z_1, \dots, Z_n$, and let $X_n = \sum_{k=1}^n Z_k$.

The following result, that shows the tight connection between uniform stick breaking and sampling of exponential random variables, can be found in [7].

Lemma 3. (Sukhatme [22])

$$(Z_1, Z_2, \dots, Z_n) \stackrel{d}{=} (Y_n, Y_{n-1}, \dots, Y_1) \tag{7}$$

where $Y_k = (n-k+1)\left(Z_{(k-2)} - Z_{(k-1)}\right)$ *for* $1 \leq k < n$ *and* $Y_n = nZ_{(1)}$.

Let $S_n^* = \max\{S_1, \dots, S_n\}$ denote the largest piece. The following is then an immediate consequence of Lemma 3.

Lemma 4.

$$S_n^* \stackrel{d}{=} \sum_{i=1}^n \frac{S_i}{i}. \tag{8}$$

Proof. We need the well known property of uniform spacings that

$$(S_1, \dots, S_n) \stackrel{d}{=} (Z_1/X_n, \dots, Z_n/X_n). \tag{9}$$

It then follows from Lemma 3 that

$$\sum_{i=1}^n \frac{S_i}{i} \stackrel{d}{=} \sum_{i=1}^n \frac{Z_i/X_n}{i} = \frac{\sum_{i=1}^n \frac{Z_i}{i}}{X_n} \stackrel{d}{=} \frac{\sum_{i=1}^n \frac{Y_{n-i+1}}{i}}{X_n} \stackrel{d}{=} \frac{Z_{(n)}}{X_n} \stackrel{d}{=} S_n^*. \quad \square$$

Now consider an M/D/1 processor sharing queue (see [19]), where customers arrive according to a Poisson process $(N_t)_{t>0}$ at times $(\tau_n)_{n\in\mathbb{N}}$. All customers have a deterministic service requirement x. When there are n customers in the system, each customer is served with rate $1/n$. Consider the already finished work V_t at time t of the first customer (the customer entering an empty system). Let κ_x denote the total sojourn time of the first customer in the system, which is given by the first hitting time of V_t of the level x.

Theorem 5.

$$\kappa_x \stackrel{d}{=} H_x. \tag{10}$$

Proof. Since the server equally distributes its capacity we have

$$dV_t = \frac{1}{N_t + 1}\, dt$$

for $\tau_i < t < \tau_{i+1}$. Thus

$$V_t = \int_0^t \frac{1}{N_s + 1}\, ds = \sum_{i=1}^{N_t} \frac{Z_i}{i} + \frac{t - \tau_{N_t}}{N_t + 1}, \tag{11}$$

and $\tau_{N_t} \stackrel{d}{=} X_{N_t}$. We know that, given $N_t = n$,

$$(S_1, S_2, \ldots, S_{n+1}) \stackrel{d}{=} (Z_1, Z_2, \ldots, Z_n, t - Z_n) \tag{12}$$

where the S_k are the lengths of the pieces of a stick of length t broken randomly. From (8), (11) and (12) we obtain

$$V_t \stackrel{d}{=} \sum_{i=1}^{N_t+1} \frac{S_i}{i} \stackrel{d}{=} S^*_{N_t+1} \tag{13}$$

Let $M_t = \max_{0\le s\le t} A_s$ be the running maximum of the renewal age process. Given that $N_t = n$ it follows that M_t is equal to $\max\{Z_1, Z_2, \ldots, Z_n, t - Z_n\}$. Using the property (12) again it follows that M_t has the same distribution as V_t. Since both processes are increasing, we conclude that for all $t \ge 0$

$$\mathbb{P}(\kappa_x \le t) = \mathbb{P}(V_t \ge x) = \mathbb{P}(M_t \ge x) = \mathbb{P}(H_x \le t),$$

proving the result. □

In [8] a derivation of the Laplace-Stieltjes transform for κ_x is given using a connection to the *Yule process*, leading to the geometric representation of τ_x. A similar result is obtained in [18] for the more general class of $M/G/1$ queues with symmetric service disciplines (of which processor sharing is a special case).

3 Further properties of the random variables

In Section 2 we have introduced the random variables H_x, T_x and R_x that are associated with the hitting time of the renewal age process. In this section we derive various properties for these random variables, including stochastic recursive equations, scaling properties, and three characterizations of the distribution functions.

3.1 Stochastic recursive equations

Lemma 6. *We have*

$$R_x \stackrel{d}{=} Z + \mathbb{1}_{\{Z<x\}} R_x, \tag{14}$$

$$T_x \stackrel{d}{=} \mathbb{1}_{\{Z<x\}} (T_x + Z), \tag{15}$$

$$H_x \stackrel{d}{=} \mathbb{1}_{\{Z\geq x\}} x + \mathbb{1}_{\{Z<x\}} (H_x + Z), \tag{16}$$

where Z denotes an independent exponential random variable.

Proof. If $Z_1 < x$ then $R_x = Z_1$, else $R_x = Z_1 + R^*$ where $R^* \stackrel{d}{=} R_x$ is independent of Z_1, thus (14) follows. A similar argument leads to (15), and the third relation follows from (15) and $H_x = T_x + x$. □

We rewrite the relations in Lemma 6 into the stochastic recursive equations

$$R_x^n \stackrel{d}{=} Z_n + \mathbb{1}_{\{Z_n<x\}} R_x^{n-1}, \tag{17}$$

$$T_x^n \stackrel{d}{=} \mathbb{1}_{\{Z_n<x\}} (T_x^{n-1} + Z_n), \tag{18}$$

$$H_x^n \stackrel{d}{=} \mathbb{1}_{\{Z_n\geq x\}} x + \mathbb{1}_{\{Z_n<x\}} (H_x^{n-1} + Z_n). \tag{19}$$

Then R_x, T_x and H_x can be seen as limiting variables of the Markov chains $(R_x^n)_{n\in\mathbb{N}}$, $(T_x^n)_{n\in\mathbb{N}}$ and $(H_x^n)_{n\in\mathbb{N}}$ on the state space $[0,\infty)$.

Lemma 7. *We have that*

$$(R_x^n, T_x^n, H_x^n) \xrightarrow{d} (R_x, T_x, H_x).$$

Proof. From (18) we obtain

$$\Phi_{T_x^n}(s) = \mathbb{E}(\mathbb{1}_{\{Z<x\}} e^{-sZ}) \Phi_{T_x^{n-1}}(s),$$

where Z is some independent exponential random variable. Since

$$\Phi_{T_x}(s) = \mathbb{E}(\mathbb{1}_{\{Z<x\}} e^{-sZ}) \Phi_{T_x}(s),$$

it follows that

$$\begin{aligned} \left|\Phi_{T_x^n}(s) - \Phi_{T_x}(s)\right| &= \mathbb{E}(\mathbb{1}_{\{Z<x\}} e^{-sZ}) \left|\Phi_{T_x^{n-1}}(s) - \Phi_{T_x}(s)\right| \\ &= \mathbb{E}(\mathbb{1}_{\{Z<x\}} e^{-sZ})^{n-1} \left|\Phi_{T_x^1}(s) - \Phi_{T_x}(s)\right|, \end{aligned}$$

so that $\left|\Phi_{T_x^n}(s) - \Phi_{T_x}(s)\right| \to 0$. The other assertions can be proved similarly. □

Consider again the renewal process, but now inserting additional renewal epochs whenever the age process $(A_t)_{t\geq 0}$ passes the level x. In doing so we form a new renewal process with truncated exponential epochs. Let $(\widetilde{A}_t)_{t\geq 0}$ be the age process of the new renewal process. Then T_x^n can be interpreted as the time since the last additional renewal epoch was inserted.

3.2 Scaling properties

In this section we write $R_{x,\lambda}$, $T_{x,\lambda}$ and $H_{x,\lambda}$ instead of R_x, T_x and H_x to stress the dependence on both x and λ. Again we let Z denote an independent exponentially distributed random variable with mean $1/\lambda$.

Let K denote a geometric random variable with $\mathbb{P}(K = 1) = q$ and let A_i denote i.i.d. random variables with $\lambda = \mathbb{E}A_i < \infty$. Rényi's theorem for geometric sums (see, for instance, [3, 17]) states that, as $q \to 0$,

$$q \sum_{k=1}^{K} A_i \xrightarrow{d} Z. \tag{20}$$

In our situation $T_{x,\lambda}$ is a geometric sum where the summands depend on the parameter $q = e^{-\lambda x}$, so that Rényi's theorem is not applicable. Kalashnikov [17] has generalized the theorem for the case where the A_i depend on the parameter q, showing that if

$$\lim_{q \to 0} \int_{\varepsilon/q}^{\infty} \mathbb{P}(A_1 > u)\, du \to 0 \tag{21}$$

for all $\varepsilon > 0$ then (20) remains true. For our geometric sum $T_{x,\lambda}$ condition (21) is clearly satisfied since A_1 is a truncated exponential random variable and $\varepsilon/q > x$ if x is sufficiently large. We give a short and stand-alone proof for this.

Theorem 8 (Kalashnikov [17])**.**

$$\frac{R_{x,\lambda}}{e^{\lambda x}} \xrightarrow{d} Z. \tag{22}$$

as $\lambda x \to \infty$. The same is true for $T_{x,\lambda}$ and $H_{x,\lambda}$.

Proof. We have

$$\Phi_{R_x}(\lambda s/e^{\lambda x}) = \frac{\lambda}{\lambda + \lambda s \cdot \exp\left(\lambda x s e^{-\lambda x}\right)} \to \frac{1}{1+s}.$$

Since $\lambda e^{-\lambda x} = \lambda x e^{-\lambda x}/x \to 0$ we get $\lambda/(\lambda + \lambda s e^{-\lambda x}) \to 1$. Using $\lambda Z/e^{\lambda x} \to 0$, we conclude from (4) that $e^{-\lambda x} T_{x,\lambda} \xrightarrow{d} Z$ and $e^{-\lambda x} H_{x,\lambda} \xrightarrow{d} Z$. □

We now state an interesting scaling property that leads to limit results for the case that λx converges to some finite number.

Proposition 9. *For all $c \in \mathbb{R}^+$,*

$$\frac{R_{x,\lambda}}{c} \stackrel{d}{=} R_{\frac{x}{c},\lambda c}. \tag{23}$$

Proof. The results follows immediately from

$$\Phi_{R_x}(s/c) = \frac{\lambda}{\lambda + \frac{s}{c} e^{(\lambda + \frac{s}{c})x}} = \frac{c\lambda}{c\lambda + s e^{(c\lambda + s)\frac{x}{c}}}.$$

□

Theorem 10. *If $\lambda x \to \gamma \in (0, \infty)$ then*

$$\left(\frac{R_{x,\lambda}}{x}, \frac{T_{x,\lambda}}{x}, \frac{H_{x,\lambda}}{x} \right) \xrightarrow{d} \left(R_{1,\gamma}, T_{1,\gamma}, H_{1,\gamma} \right).$$

Proof. Follows immediately from (23), since $\frac{R_{x,\lambda}}{x} \stackrel{d}{=} R_{1,\lambda x}$. □

3.3 Distribution function I

From the geometric sum representation (3) we immediately get a characterization for the distribution function of H_x in terms of infinite convolutions of the distribution of B. In this subsection, and the next two subsections, we show that more explicit characterizations can be obtained. In Subsection 2.2 we proved an equivalence result for the renewal age process and the M/D/1 processor sharing queue. We now use this connection to obtain a first characterization of the distribution function of H_x.

The distribution of the largest piece S_n^*, also known as maximal uniform spacing, is given by Whitworth's formula (see [7, 13])

$$\mathbb{P}(S_n^* \leq x) = t^{1-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \, (t-kx)_+^{n-1} \,, \tag{24}$$

where $x_+ = \max\{0, x\}$. It is readily seen from (9) that the conditional distribution of the running maximum M_t given that $N_t = n$ is the same as the distribution of S_{n+1}^*, and hence

$$M_t \stackrel{d}{=} S_{N_t+1}^*.$$

Using $\mathbb{P}(H_x > t) = \mathbb{P}(M_t \leq x)$ and conditioning on the number of events in $[0, t]$ leads to a first representation of the distribution of H_x.

Theorem 11. *For $t \geq 0$,*

$$\overline{F}_{H_x}(t) = e^{-\lambda t} \mathbb{1}_{\{t \leq x\}} + e^{-\lambda t} \sum_{n=1}^{\infty} \sum_{k=0}^{n+1} \frac{(\lambda t)^n}{n!} \binom{n+1}{k} (-1)^k t^{-n} \, (t-kx)_+^n \,. \tag{25}$$

Remark 12. Formula (24) has another interpretation. Locate n points randomly on a circle with circumference t. We can represent the gaps between the points on the circle by uniformly chosen random variables S_i, $i = 1, \ldots, n-1$ in $[0, t]$. Attach to each random point an arc of length x such that the point lies in the middle of the arc. The probability that the circle is completely covered by the arcs is given by

$$\mathbb{P}(\max_{i=1,\ldots,n} S_i \leq x) = \mathbb{P}(S_n^* \leq x),$$

(see [15], or [12], Theorem I.2). In this connection Whitworth's formula (24) is sometimes called Steven's formula (see [14]).

3.4 Distribution function II

We now present explicit characterizations for the distribution functions of R_x, T_x and H_x in terms of finite series. The proof is purely analytical and builds upon earlier work of Bellman and Cooke [2] on differential-difference equations.

Theorem 13. *We have*

$$\overline{F}_{R_x}(t) = \sum_{j=0}^{\lfloor t/x \rfloor} (-\lambda e^{-\lambda x})^j \frac{(t-jx)^j}{j!}, \quad t \geq x, \tag{26}$$

with $\lfloor a \rfloor$ the integer part of a. Moreover

$$\overline{F}_{H_x}(t) = \overline{F}_{R_x}(t) - e^{-\lambda x} \overline{F}_{R_x}(t-x), \quad t \geq x, \tag{27}$$

and $\overline{F}_{T_x}(t) = \overline{F}_{H_x}(t+x)$, $t \geq 0$.

Proof. Denote the right-hand side of (26) by $\omega(t)$, which is continuous for $t > 0$ and continuously differentiable for $t > 0$ except at $t = x$; at $t = x$ we have $\omega'(x-) = 0$, $\omega'(x+) = -\lambda e^{-\lambda x}$. We also note that $\omega(t) = 1$ for $0 \leq t \leq x$ and that $\omega(t) = 0$ for $t < 0$. One readily obtains that for $t > x$, t not a multiple of x,

$$\omega'(t) = -\lambda e^{-\lambda x}\omega(t - x), \tag{28}$$

with initial condition $\omega(t) = 1$, $0 \leq t \leq x$. Equation (28) is also valid when t is a multiple of x by continuity of $\omega'(t)$ for $t > x$. The equation (28) is a differential-difference equation (see [2], Chapter 3). According to the theory in [2], Sections 3.7 and 4.4), we have to consider the roots s of the characteristic equation

$$s + \lambda e^{-(\lambda+s)x} = 0. \tag{29}$$

Note that these roots are in fact the singularities of $\Phi_{H_x}(s)$. In the appendix we show that all characteristic roots lie in the half-plane $\mathrm{Re}(s) < 0$. By [2], Corollary 4.2 on p. 115 it follows that $\overline{F}_{R_x}(t) \to 0$ as $t \to \infty$. Hence, the Laplace transform $\omega(s) = \int_0^\infty e^{-st}\overline{F}_{R_x}(t)\ dt$ is well-defined and analytic in $\mathrm{Re}(s) > 0$. By direct calculation from (26) or (28) we get

$$\omega(s) = \frac{1}{s + \lambda e^{-(\lambda+s)x}}, \quad \mathrm{Re}(s) > 0. \tag{30}$$

A computation then shows that the Laplace transform of $\overline{F}_{R_x}(t) - e^{-\lambda x}\overline{F}_{R_x}(t - x)$ is given by

$$\frac{1 - e^{-(\lambda+s)x}}{s + \lambda e^{-(\lambda+s)x}}, \quad \mathrm{Re}(s) > 0. \tag{31}$$

Finally, we note that the ordinary Laplace transform of $\overline{F}_{H_x}(t)$ is equal to $s^{-1}(1-\mathbb{E}e^{-sH_x})$, and by (6) it is seen that $\overline{F}_{H_x}(t)$ and $\overline{F}_{R_x}(t) - e^{-\lambda x}\overline{F}_{R_x}(t - x)$ have the same Laplace transform. The proof is completed by [2], Theorem 1.1 on p. 7. $\square$

Equation (27), which can be written as

$$F_{H_x}(t) = F_{R_x}(t) + e^{-\lambda x}\overline{F}_{R_x}(t - x), \quad t \geq x, \tag{32}$$

expresses an interesting relation between the random variables H_x and R_x. This relation can be derived probabilistically. Let $M_t = \max_{0 \leq s \leq t} A_t$ be the running maximum of the renewal age process, i.e., $H_x = \inf\{s > 0 | M_s = x\}$. Since $H_x \leq t$ occurs either if the process hits x before N_t or if it hits x after N_t and there are no epochs ending during $[t - x, t]$, we obtain

$$\mathbb{P}(H_x \leq t) = \mathbb{P}(M_{N_t} \geq x) + \mathbb{P}(M_{N_t} < x, N_t - N_{t-x} = 0). \tag{33}$$

It follows from the properties of Poisson processes and $\{M_{N_t} < x, N_t - N_{t-x} = 0\} = \{M_{N_{t-x}} < x, N_t - N_{t-x} = 0\}$ that

$$\mathbb{P}(M_{N_t} < x, N_t - N_{t-x} = 0) = \mathbb{P}(M_{N_{t-x}} < x)\mathbb{P}(N_t - N_{t-x} = 0).$$

Since $\mathbb{P}(N_t - N_{t-x} = 0) = e^{-\lambda x}$ and $\{M_{N_t} \geq x\} = \{R_x \leq t\}$ we get (32).

Remark 14. We can also obtain the inverse of $\omega(s)$ by writing

$$\Phi(s) = \sum_{j=0}^{\infty} (-\lambda e^{-\lambda x})^j e^{-sxj} s^{-(j+1)}, \tag{34}$$

provided $\mathrm{Re}(s)$ is sufficiently large. Then, using

$$\int_0^{\infty} (t - jx)_+^j e^{-st} dt = \frac{e^{-sjx} j!}{s^{j+1}} \tag{35}$$

and the uniqueness of the Laplace transform, we arrive at (26). For the formal arguments that go with this approach we refer to [2], Section 4.7.

3.5 Distribution function III

In this section we represent the distribution functions of R_x, T_x and H_x as series that involve the infinitely many poles of Φ_{R_x}. These poles are in fact the complex roots of

$$se^{sx} = -\lambda e^{-\lambda x} \tag{36}$$

and can be expressed in terms of the branches of the Lambert W function (see Section 4). Equation (36) has two real roots s_* and s_0 on the negative real axis which we order as $s_0 \leq s_* < 0$. In Section 4.1 we prove the following result.

Lemma 15. *For $\lambda x < 1$ we have $s_* = -\lambda$. For $\lambda x > 1$ we have $s_0 = -\lambda$ and*

$$s_* = -\frac{1}{x} \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} \left(\lambda x e^{-\lambda x}\right)^n. \tag{37}$$

For $\lambda x = 1$ we have $s_* = s_0 = -\lambda$.

For the case $\lambda x < 1$ a simple characterization of s_0 is not available (see [5], Section 2.4), but s_0 can be determined numerically using the method of Newton-Raphson. The root s_* of (36) is the only one lying in the closed disk $|s| \leq 1/x$. All other roots lie in the half-plane $\mathrm{Re}(s) \leq s_0$, and, more particular, on the set

$$\{s \in \mathbb{C} : \mathrm{Re}(s) \leq s_0\ ,\ |se^{sx}| = \lambda e^{-\lambda x}\}. \tag{38}$$

We order the roots s_k, $k \in \mathbb{Z}$ lying on (38) in conjugate pairs $\bar{s}_k = s_{-k}$ and according to the value of $\arg(se^{sx}) = \arg(s) + x\mathrm{Im}(s)$, so that

$$\arg(s_k) + x\ \mathrm{Im}(s_k) = 2\pi(k + 1/2).$$

In this way, the roots are $s_0, s_1, s_2, \ldots$ are arranged in order of decreasing real parts.

The following lemma will be shown in Section 4.2.

Lemma 16. *The roots s_k with $|s_k| > 1/x$ are obtained in the form*

$$s_k = -\frac{r_k}{x} e^{-i\psi_k} \quad ; \quad s_{-k} = \overline{s}_k \quad , \quad k = 0, 1, \ldots, \tag{39}$$

where r_k and ψ_k simultaneously satisfy (with $d = \lambda x e^{-\lambda x}$)

$$r_k \cos\psi_k - \ln(r_k/d) \;=\; 0, \tag{40}$$

$$r_k \sin\psi_k - \psi_k - 2k\pi \;=\; 0, \tag{41}$$

and where we restrict to $r_k \geq x|s_0|$, $\psi_k \in [0, \pi/2)$.

Lemma 16 gives a complete characterization of all complex roots, where each root is described in terms of the two equations (40) and (41). For solving these equations, a highly efficient procedure using two-dimensional Newton-Raphson is given in Subsection 4.3. Lemma 16 is used in Subsection 3.6 to obtain asymptotic information on the roots.

Theorem 17. *Assume $\lambda x \neq 1$. We have*

$$\overline{F}_{R_x}(t) = \frac{e^{s_* t}}{1 + x s_*} + \sum_{k=-\infty}^{\infty} \frac{e^{s_k t}}{1 + x s_k}, \tag{42}$$

and

$$\overline{F}_{H_x}(t) = \frac{\lambda + s_*}{\lambda(1 + x s_*)} e^{s_* t} + \sum_{k=-\infty}^{\infty} \frac{\lambda + s_k}{\lambda(1 + x s_k)} e^{s_k t}, \tag{43}$$

with absolute convergence when $t > x$.

Proof. By the Laplace inversion formula we have for $b > s_*$ that

$$\overline{F}_{R_x}(t) = \frac{1}{2\pi i} \lim_{B \to \infty} \int_{b-iB}^{b+iB} \frac{e^{st}}{s + \lambda e^{-(\lambda+s)x}} ds. \tag{44}$$

Shifting the integration contour $\mathrm{Re}(s) = b$ further and further to the left yields a series representation for $\overline{F}_{R_x}(t)$ involving all roots s_k of (36). We refer to [2], Sections 4.1 and 4.2, where this process of shifting the integration path to the left is treated rigourously. $\square$

Lemma 18. *For $\lambda x = 1$ we have that*

$$\overline{F}_{R_x}(t) = \frac{2t}{x} e^{-t/x} + \sum_{k \neq 0} \frac{e^{s_k t}}{1 + x s_k}, \quad t > x, \tag{45}$$

and subsequently,

$$\overline{F}_{H_x}(t) = 2e^{-t/x} + \sum_{k \neq 0} e^{s_k t}, \quad t > x. \tag{46}$$

Proof. In case $\lambda x = 1$, we have that $s_* = s_0 = -\lambda$. Then, the integrand in (44) has a second order pole at $s = s_* = s_0 = 1/x$ with residue $2(t/x)\exp(-t/x)$. This yields (45) and (46). An alternative derivation is by taking the limits $\lambda \uparrow 1/x$ and/or $\lambda \downarrow 1/x$ in equations (42) and (43), which requires an analysis of the two roots s_* and s_0 when $\lambda x \to 1$. $\square$

3.6 Asymptotics for the tail distribution

We obtain from (43) the asymptotic expression

$$\overline{F}_{H_x}(t) \sim \frac{\lambda + s_*}{\lambda(1 + x s_*)} e^{s_* t} + \frac{\lambda + s_0}{\lambda(1 + x s_0)} e^{s_0 t}, \quad t \to \infty, \tag{47}$$

which can serve as an approximation to $\overline{F}_{H_x}(t)$ for larger values of t. Note that the term involving s_* vanishes when $\lambda x < 1$ and the term involving s_0 vanishes when $\lambda x > 1$, and that (47) is consistent with (46) for $\lambda x = 1$.

From (40) and (41) it is not hard to get asymptotic information on r_k and ψ_k as $k \to \infty$.

Lemma 19. *We have*

$$r_k = q_k - \frac{\ln(q_k/d)}{q_k} + \frac{\ln^2(q_k/d)}{2q_k} + O\left(\frac{\ln^4(q_k/d)}{q_k^3}\right), \tag{48}$$

$$\psi_k = \frac{\pi}{2} - \frac{\ln(q_k/d)}{q_k} + O\left(\frac{\ln^3(q_k/d)}{q_k^3}\right). \tag{49}$$

in which $q_k = 2\pi(k+1/4)$ *and* $d = \lambda x e^{-\lambda x}$*.*

Proof. We have from (40) and (41) that for $k = 1, 2, \ldots$

$$r_k \geq 2\pi k \quad , \quad \psi_k \geq \arccos\left[\frac{1}{2\pi k}\ln(2\pi k/d)\right], \tag{50}$$

since $r^{-1}\ln(r/d)$ decreases in $r \geq d \cdot e \in (0,1)$. Some further iterations with (40) and (41) yield the result. □

Lemma 19 shows that s_k grows like $2\pi(k+1/4)$. Then, from

$$e^{s_k x} = \frac{-\lambda e^{-\lambda x}}{s_k}, \tag{51}$$

we conclude that there is a decay of the terms in the series at the right-hand sides of (43) and (47) like $k^{-t/x}$. This rapid decrease of the higher-order terms makes (47) a highly accurate approximation, even for moderate values of t. Some numerical evidence for this statement is presented in Tables 1 and 2, in which we denote by $(43)_j$ the approximation obtained from (43) by including the terms $k = -j, -j+1, \ldots, j$.

t	$\overline{F}_{H_x}(t)$	(47)	$(43)_1$	$(43)_2$
1	3.0301e-001	4.2424e-001	3.7279e-001	3.1492e-001
2	4.4709e-002	4.5769e-002	4.4073e-002	4.4678e-002
3	4.9883e-003	4.9378e-003	4.9936e-003	4.9882e-003
4	5.3264e-004	5.3271e-004	5.3260e-004	5.3264e-004
5	5.7444e-005	5.7471e-005	5.7444e-005	5.7444e-005
6	6.2007e-006	6.2002e-006	6.2007e-006	6.2007e-006
7	6.6891e-007	6.6890e-007	6.6891e-007	6.6891e-007
8	7.2164e-008	7.2164e-008	7.2164e-008	7.2164e-008
9	7.7853e-009	7.7853e-009	7.7853e-009	7.7853e-009
10	8.3992e-010	8.3992e-010	8.3992e-010	8.3992e-010

Table 1: Results for $\overline{F}_{H_x}(t)$ when $\lambda = 0.45$, $x = 0.9$.

Note that (47) is consistent with (22). To see this, first observe from (37) that $s_* \sim -\lambda e^{-\lambda x}$ as $\lambda x \to \infty$. Hence, (47) yields

$$\overline{F}_{H_x}(\tfrac{1}{\lambda}e^{\lambda x}t) \sim \frac{\lambda + s_*}{\lambda(1+xs_*)} e^{s_* \frac{1}{\lambda} e^{\lambda x} t} \sim e^{-t}, \quad \lambda x \to \infty. \tag{52}$$

For R_x we obtain from (42) the asymptotic expression (for $\lambda x \neq 1$)

$$\overline{F}_{R_x}(t) \sim \frac{e^{s_* t}}{1 + xs_*}. \tag{53}$$

In [18], Proposition 1, a related result is presented for the more general class of symmetric $M/G/1$ queues. For $\lambda x > 1$ (53) sharpens the result in [18].

t	$\overline{F}_{H_x}(t)$	(47)	$(43)_1$	$(43)_2$
1	7.6648e-001	7.9871e-001	7.8759e-001	7.7073e-001
2	4.2690e-001	4.2734e-001	4.2667e-001	4.2688e-001
3	2.2867e-001	2.2865e-001	2.2867e-001	2.2867e-001
4	1.2233e-001	1.2233e-001	1.2233e-001	1.2233e-001
5	6.5454e-002	6.5454e-002	6.5454e-002	6.5454e-002
6	3.5020e-002	3.5020e-002	3.5020e-002	3.5020e-002
7	1.8737e-002	1.8737e-002	1.8737e-002	1.8737e-002
8	1.0025e-002	1.0025e-002	1.0025e-002	1.0025e-002
9	5.3638e-003	5.3638e-003	5.3638e-003	5.3638e-003
10	2.8699e-003	2.8699e-003	2.8699e-003	2.8699e-003

Table 2: Results for $\overline{F}_{H_x}(t)$ when $\lambda = 1.8$, $x = 0.9$.

3.7 M/D/1 first-come-first-served queue

A.K. Erlang's 1909 paper [10] introducing the M/D/1 queue is generally considered to be the starting point of queueing theory. For Poisson arrivals with rate λ, deterministic service requirements x, and first-come-first-served, Erlang's results on the stationary waiting time W reads (assuming $\lambda x < 1$ for stability)

$$F_W(t) = (1 - \lambda x)e^{\lambda t} \sum_{j=0}^{\lfloor t/x \rfloor} (-\lambda e^{-\lambda x})^j \frac{(t - jx)^j}{j!}, \quad t \geq 0. \tag{54}$$

Hence, the waiting time distribution can be expressed in terms of our function $\overline{F}_{R_x}$ as $F_W(t) = (1 - \lambda x)e^{\lambda t}\overline{F}_{R_x}(t)$. Using (42) and $s_* = -\lambda$ (since $\lambda x < 1$) then yields

$$\overline{F}_W(t) = \sum_{k=-\infty}^{\infty} \frac{\lambda x - 1}{1 + x s_k} e^{(\lambda + s_k)t}. \tag{55}$$

It is remarkable that the approximation

$$\overline{F}_W(t) \approx \frac{\lambda x - 1}{1 + x s_0} e^{(\lambda + s_0)t}, \tag{56}$$

for large values of t, was already stated (without proof) in Erlang's paper [10] (see also [20], p. 54).

Remark 20. The probability p_n that n customers are served during a busy period of the M/D/1 queue ($\lambda x < 1$) is given by (already found by Borel in 1942 [4])

$$p_n = \frac{1}{\lambda x} \frac{n^{n-1}}{n!} (\lambda x e^{-\lambda x})^n.$$

This confirms the fact that formula (37) is valid for $\lambda x < 1$, since $s_* = -\sum_{n=1}^{\infty} \lambda p_n = -\lambda$.

4 Analysis of the characteristic roots

We consider for positive λ and x the roots s of the equation

$$s e^{sx} = -\lambda e^{-\lambda x}, \tag{57}$$

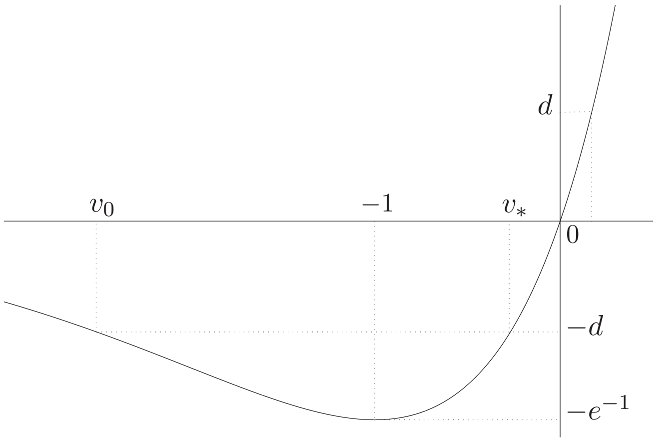

Figure 2: Plot of $v \mapsto ve^v$.

which are required at several places in the main text. Using $v = sx$ and $\mu = \lambda x$ simplifies (57) to

$$ve^v = -\mu e^{-\mu}. \tag{58}$$

The multi-valued inverse of the function $v \mapsto ve^v$ has a long history in mathematics. It is treated as one of the key examples in the book of De Bruijn [5] (Sections 2.3-4), and since the overview paper of Corless et al. [6] it is known as the Lambert W function. The analysis presented in this section can in part be found in [2], Section 12.7 on pp. 406-10, where the emphasis lies on roots of (57) of large modulus. The series representations in Section 3.5, which we use to approximate $\overline{F}_{H_x}(t)$, let us focus more on the roots of small(er) modulus, and how to compute them.

4.1 Proof of Lemma 15

Let $d = \mu e^{-\mu}$, $c = \mu - \ln \mu$; $d = e^{-c}$. We have $c \geq 1$, $d \leq e^{-1}$ with equality if and only if $\mu = 1$. We let v_* and v_0 be the two real roots of (58), with $v_0 \leq -1 \leq v_* < 0$, see Figure 2 ($v_* = v_0 = -1$ if and only if $d = 1/e$, i.e., $\mu = 1$).

Clearly, the roots v of (58) all lie in the set

$$\{v \in \mathbb{C} : |ve^v| = d\}. \tag{59}$$

With reference to Subsection 4.2 and in particular Figure 3, we note that this set consists of two parts, viz. a part contained in the unit disk $|v| \leq 1$ and a part contained in the set $|v| \geq 1$ (in [2], Section 12.7 attention is limited to the latter set). We consider first the part contained in $|v| \leq 1$. The mapping $v \mapsto ve^v$ is invertible around $v = 0$, with the inverse given by Lagrange's theorem as (see [5], Sec. 2.3)

$$ve^v = w \quad ; \quad v(w) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{n-1}}{n!} w^n, \tag{60}$$

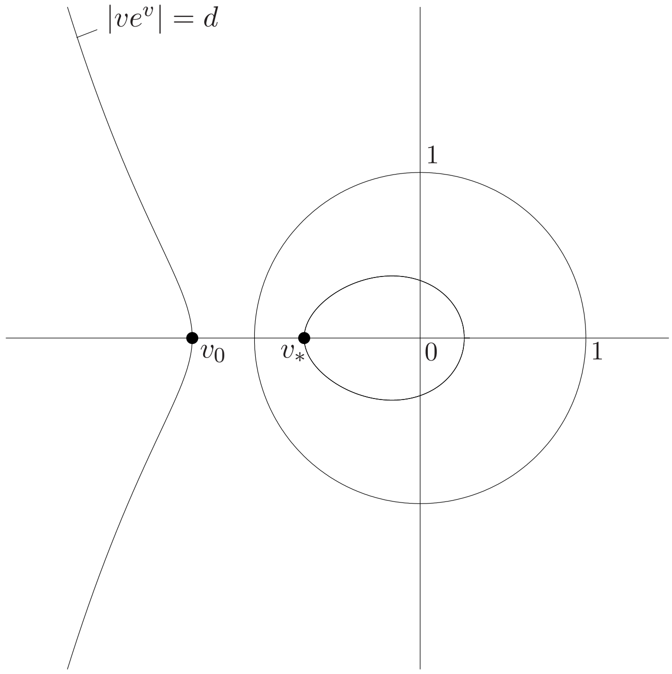

Figure 3: The roots of $ve^v = -\mu e^{-\mu}$ for $\mu = 0.7$. The two real roots are $v_0 = -1.3755$ and $v_* = -0.7000$. Other roots are $v_1 = -3.1475 + 7.4545i$, $v_2 = -3.7215 + 13.8751i$, $v_3 = -4.0834 + 20.2211i$ and $v_4 = -4.3486 + 26.5411i$.

and where by Stirling's formula the series converges absolutely in $|w| \leq 1/e$. Accordingly,

$$v(-d) = v_* = -\sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} d^n \tag{61}$$

is the only root of (58) in the disk $|v| \leq |v_*|$. The set $\{v(de^{i\alpha}) : \alpha \in [0, 2\pi)\}$, which is obviously contained in $|v| \leq |v_*|$ by (60) and (61), coincides with the part of the set in (59) contained in the unit disk.

4.2 Proof of Lemma 16

We now also consider points v of the set (59) with $|v| \geq |v_*|$, and we first consider the case that $\mu \neq 1$. We note that for any $a \in \mathbb{R}$ the function

$$b \geq 0 \mapsto |(a + ib)e^{a+ib}| = e^a (a^2 + b^2)^{1/2} \tag{62}$$

is strictly increasing. Therefore, see Figure 2, the equation $|(a + ib)e^{a+ib}| = d$ has: no solution $b \geq 0$ when $a > v(d)$, one solution $b \geq 0$ when $v_* = v(-d) \leq a \leq v(d)$, no solution $b \geq 0$ when $v_0 < a < v_*$, one solution $b \geq 0$ when $a \leq v_0$. Also observe that the set in (59) is symmetric with respect to the real axis.

We shall now consider in more detail the roots of (58) that lie in $\mathrm{Re}(v) \leq v_0$, i.e., that are located on the curve depicted in Figure 3 that crosses the real axis at $a = v_0$. In terms of polar coordinates $v = re^{i\varphi}$, we have for this curve

$$ve^v = re^{r\cos\varphi}e^{i\varphi + ir\sin\varphi} \tag{63}$$

in which we choose $\varphi \in (\pi/2, 3\pi/2)$. There is thus the parametrization

$$\ln r + r\cos\varphi = \ln d = -c \quad , \quad r \geq |v_0| \ , \ \varphi \in (\pi/2, 3\pi/2) \tag{64}$$

of the considered curve.

We shall now analyze this parametrization somewhat further. To that end let $\psi = \pi - \varphi \in [0, \pi/2)$, and write (64) as

$$r\cos\psi = \ln(r/d) \quad , \quad r \geq |v_0| \ , \ \psi \in [0, \pi/2). \tag{65}$$

Here we have restricted to $\psi \geq 0$ for reasons of symmetry. It is easy to see that (65) has for any $\psi \in [0, \pi/2)$ exactly one solution $r(\psi) \geq |v_0| > 1$ that increases from $|v_0|$ at $\psi = 0$ to ∞ as $\psi \uparrow \pi/2$. We also compute (with $v = -re^{-i\psi}$)

$$\arg(ve^v) = \pi - \psi + r\sin\psi. \tag{66}$$

Lemma 21. *The argument* $\arg(ve^v)$ *increases in* $\psi \in [0, \pi/2)$.

Proof. From (65) we have

$$r'(\psi) = \frac{r(\psi)\sin\psi}{\cos\psi - 1/r(\psi)}. \tag{67}$$

Therefore,

$$\begin{aligned} \frac{d}{d\psi}[\pi - \psi + r\sin\psi] &= -1 + r'(\psi)\sin\psi + r(\psi)\cos\psi \\ &= -1 + \frac{r(\psi)\sin^2\psi}{\cos\psi - 1/r(\psi)} + r(\psi)\cos\psi \\ &= \frac{1/r(\psi) + r(\psi) - 2\cos\psi}{\cos\psi - 1/r(\psi)} \ > \ 0, \end{aligned} \tag{68}$$

where we have used that

$$r(\psi)\cos\psi \ = \ \ln(r(\psi)/d) \ \geq \ \ln(r(0)/d) \ = \ r(0) \ = \ |v_0| \ > \ 1, \tag{69}$$

so that $1/r(\psi) + r(\psi) - 2\cos\psi > 0$ and $\cos\psi - 1/r(\psi) > 0$. Hence, $\arg(ve^v)$ indeed increases in $\psi \in [0, \pi/2)$ and does so from π at $\psi = 0$ to ∞ as $\psi \uparrow \pi/2$. □

We conclude from Lemma 21 that for $k = 0, 1, \ldots$ there is a unique $\psi_k \in [0, \pi/2)$ such that $\pi - \psi_k + r(\psi_k)\sin\psi_k = 2k\pi + \pi$.

Until now we have assumed that $\mu \neq 1$. In the case that $\mu = 1$, we have that $v_0 = v_* = -1$ and $d = 1/e$. The two parts of the set (59) inside and outside the unit disk now meet at the point $v = -1$ and together constitute the well-known Szegö curve (see [23] and [16]), mirrored about the imaginary axis. However, for the analysis of the roots outside the unit disk, the developments just given for the case $\mu \neq 1$ remain equally valid.

This gives us Lemma 16.

4.3 Computation of the roots

The s_k's with $k = 1, 2, \ldots$ can be computed by writing (40) and (41) in $a + ib = re^{i\psi}$ notation as

$$\underline{\kappa}(\underline{a}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \underline{a} = \begin{bmatrix} a \\ b \end{bmatrix}; \quad \underline{\kappa}(\underline{a}) = \begin{bmatrix} a - \frac{1}{2}\ln(a^2+b^2) + \ln d \\ b - \arctan(b/a) - 2\pi k\pi \end{bmatrix} \tag{70}$$

and solving (70) by using a two-dimensional Newton-Raphson iteration. Thus we iterate according to

$$\underline{a}_{n+1} = \underline{a}_n - J^{-1}(\underline{a}_n)\underline{f}(\underline{a}_n), \quad n = 0, 1, \ldots, \tag{71}$$

in which J^{-1} is the inverse of the matrix J whose entries are the partial derivatives of the two components of the vector function $\underline{\kappa}$ and that is in the present case given as

$$J(\underline{a}) = \begin{bmatrix} 1 - \frac{a}{a^2+b^2} & \frac{-b}{a^2+b^2} \\ \frac{b}{a^2+b^2} & 1 - \frac{a}{a^2+b^2} \end{bmatrix}. \tag{72}$$

It is seen that $J(\underline{a})$ is in all cases of interest close to the identity matrix, which is why the Newton-Raphson method has excellent convergence properties. A manifestation of this is the fact that we can start for any k in (70) with the same starting value $\underline{a}_0 = [0, 2\pi]^T$ (although more sophisticated starting values may be taken).

5 Acknowledgement

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