Reaction on comments by E. Geddes on our paper JAES, Vol 57(12), 2009, pp. 1004-1015

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Our principal motivation for writing this paper was the discovery of the formula in Eq. (11) that expresses the on-axis pressure, both in the near-field and the far-field, explicitly in terms of the expansion coefficients of the radially symmetric velocity profile with respect to the orthogonal Zernike circle polynomials of azimuthal order $m = 0$. With this formula in hand, one is able to estimate the expansion coefficients—and thus the profile—from on-axis, near-field pressure data. From this point onwards, one can proceed estimating acoustical quantities such as far-field, radiated power, reaction on radiator, directivity etc. It is important to emphasize here that the pressure data are collected in the near-field so that, in particular, no anechoic room is needed for the measurement. Thus, the profile retrieval method combined with the forward computation of the far-field generalizes Keele's method in the sense that also non-uniform profiles are allowed.

We agree with the commenter that there is a variety of sets of orthogonal functions on the disk that can be used to represent velocity profiles, and that one should be guided by particular practicalities in favoring one set over the other. The commenter has chosen in his book [1] the products of radial variable-scaled Bessel functions and azimuthal integer-order cosines. This is a sound and renowned orthogonal set— the foundations of this method having been laid in the late 19^{th} century, see [2, Ch. XVII, Exer. 20 on p. 381] for the history of the method with regard to the radial functions—with practical merits for loudspeaker analysis, not in the least because of validity of convenient Hankel transform relations for these functions. We regret that we did yet not compare systematically the Zernike based method with other methods, including the one that the commenter has in his book [1, Ch. 4], with respect to their merits for loudspeaker analysis. We would like to use this opportunity to say some words about this now.

The Zernike circle polynomials $Z_n^m(\rho,\varphi) = R_{|m|+2n}^{|m|}(\rho)\exp(im\varphi)$ to be considered for $0 \leq \rho \leq 1$ and $0 \leq \varphi \leq 2\pi$, are extensively used in Optical Diffraction Theory of aberrated systems. They arise naturally and uniquely among polynomial sets of orthogonal functions when certain form invariant requirements are made, see [3, 4]. Furthermore, there is the Hankel transform formula

$$\int_0^1 R_{|m|+2n}^{|m|}(\rho) J_m(u\rho)\rho d\rho = (-1)^n \frac{J_{2n+1}(u)}{u} . \quad (1)$$

This formula renders the circle polynomials a very convenient tool in both Optics, for computation of the point-spread function in the best-focus plane, and in Acoustics for computation of the far-field of flexible, baffled-piston radiators in similar manner as the commenter has this in his book in Ch. 4 for products of Bessel functions and cosines. The Hankel transform of the radial part $R_{|m|+2n}^{|m|}$ of the circle polynomial Z_n^m involves the Bessel function of order $2n + 1$, and is therefore contrary to what the commentor states, quite different from the Hankel transforms

$$\theta_n(s) = \frac{2sJ_1(s)}{s^2 - \beta_{0n}^2} \quad (2)$$

that appears in Eq. (4.2.12) and Fig. 4.4 of the commenter's book. In particular, the right-hand side of Eq. (1) above has a zero of order $2n$ at $u = 0$ while the θ_n's with $n \neq 0$ all have a zero of order 2 at $s = 0$. The Hankel transform in Eq. (1) is a very convenient starting point to express—directly or via King's formula—a variety of other acoustical quantities. This is done in our paper for the far-field and in [5] for quantities for which King's integral is instrumental. It is, at this moment, not obvious to us what could be done in this latter respect with the Hankel transforms of θ_n in Eq. (2) above.

Clearly, the Bessel-times cosine functions in [1, Ch. 4] are non-polynomial in character, and so we should indicate why we have chosen in our paper to use the circle polynomials in a different manner than by referring to [3, 4]. We are not aware of any analytic formulas for the on-axis pressure due to a non-uniform velocity profile that is expanded into orthogonal functions on the disk, other than Eq. (11) in our paper involving the expansion coefficients pertaining to the circle polynomials of azimuthal order $m = 0$. For our purposes, the validity of such a formula, especially in the near-field, is essential. The Eq. (4.1.9-10) in [1] holds in the far-field only, and would therefore not serve our purposes.

Both the system of Bessel-times-cosine functions and the system of Zernike circle polynomials can be used to represent velocity profiles and, in reverse direction, for estimating velocity profiles through expansion coefficients from far-field pressure data. For the Bessel-times-cosine functions, the latter problem has been investigated by the commenter as he writes. An important issue here is the efficiency of a particular set of orthogonal functions in representing velocity profiles in terms of magnitude and number of required coefficients. There is an abundance of analytically given profiles with explicitly known expansion into circle polynomials, see [6, Appendix A]. It is thus observed that for velocity profiles $v(\rho)$, such as $1 - \rho^2$, $\exp(-\alpha\rho^2)$, $\mathrm{sinc}(\alpha\rho)\ 2J_1(b\rho)/b\rho$, there is very fast coefficient decay, typically like $(B/n)^n$ with some B depending on the particular parameters considered in the profile. As an example, we computed the normalized coefficients

$$\frac{\int_0^1 v(\rho) N_n(\rho)\rho d\rho}{\left(\int_0^1 N_n^2(\rho)\rho d\rho\right)^{1/2}} \quad , \ n = 0, 1 \cdots \ , \tag{3}$$

of the velocity profile $v(\rho) = 1 - \rho^2$, $0 \le \rho \le 1$, for either choice

$$N_n(\rho) = J_0(\beta_{0n}\rho) \ \text{ and } R_{2n}^0(\rho) \ , \ n = 0, 1 \cdots \ . \tag{4}$$

The numbers in Eq. (3) arise as coefficients when normalized N_n, rather than the N_n themselves, are used to expand v. It is thus found that the first choice in Eq. (4) above yield

$$\frac{1}{2\sqrt{2}}\ (n=0)\ ,\ \frac{-2\sqrt{2}}{\beta_{0n}^2}\ (n=1,2\cdots)\ , \tag{5}$$

while the second choice in Eq. (4) yields

$$\frac{1}{2\sqrt{2}}\ ,\ \frac{-1}{2\sqrt{6}}\ ,\ 0,\ 0,\ \cdots\ . \tag{6}$$

Thus the coefficients in Eq. (5) above decay like $-2\sqrt{2}/\pi^2(n+1/4)^2$. While this decay is reasonably fast, it is definitely slower than the decay $(B/n)^n$ observed generally in the Zernike-based expansions for smooth profiles.

We thank the commenter for sharing his experience with us regarding the practical issues that we raised in our paper. These comments are very useful and we shall bear them in mind when continuing our investigations. Here it should be noted, however, that the polar plot, far-field retrieval method and the on-axis, near-field retrieval method may behave quite differently for one or several of these issues.

References

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