

# Acoustic holography for piston sound radiation with non-uniform velocity profiles

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The theory of orthogonal (Zernike) expansions of functions on a disk, as used in the diffraction theory of optical aberrations, is applied to obtain (semi-) analytical results for the radiation of sound due to a non-uniformly moving, baffled, circular piston. For this particular case, a scheme for retrieval of the radially symmetric part of the velocity profile from on-axis, near-field measurements on the level of expansion coefficients of the profile has been developed and demonstrated previously with simulation results and with results obtained from measured pressure data of a loudspeaker. In this contribution, (semi-) analytic expressions for transient responses, valid on the whole half-space in front of the baffle, including the axis, are presented. It thus appears that the radially symmetric part of the velocity profile on the disk is reproduced in warped form at any on-axis point as an impulse response. This offers an alternative method in present holography to estimate the velocity profile from on-axis pressure data.

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## I. INTRODUCTION

In this paper two analytic methods for estimating the radially symmetric part of a velocity profile (baffled-piston radiation) from on-axis pressure data are discussed. Both methods are based on the analytic results as developed by Zernike<sup>1</sup> and worked out by Nijboer in his 1942 thesis<sup>2</sup>; also see<sup>3</sup>. In the Nijboer-Zernike approach the exit pupil function is developed into orthogonal functions (the circle polynomials of Zernike) where the contribution to the optical point-spread function of the separate Zernike terms is available analytically (at best focus) or semi-analytically (in the focal region, see<sup>4</sup>).

The present paper discusses a method which is based on the result that any radially symmetric velocity profile  $v$  is reproduced as a time-warped impulse response corresponding to an instantaneous volume displacement of the piston at  $t = 0$  at any on-axis point. This result can be found implicitly in<sup>5</sup> (Sec. VI.B.). Thus Eq. (58) in<sup>5</sup> on transient responses shows validity of it for the special case of parabolic radiators  $v^{(n)}(\sigma) = (1 - (\sigma/a)^2)^n H(a - \sigma)$ ,  $n = 0, 1, \dots$ , with  $a$  the piston radius and  $H$  the Heaviside function, that is  $H(x) = 0, \frac{1}{2}$ , or  $1$ , according as  $x$  is negative, zero, or positive. However, the validity of this result for general, radially symmetric velocity profiles  $v(\sigma)$  does not seem to have been noticed. Also see<sup>6</sup> and<sup>7</sup> for a review of the analytical theory of acoustical transient responses.

In the sequel, the attention will be limited to radially symmetric velocity profiles. The radiated pressure is given by the Rayleigh integral, see Sec. II below. When  $v$  is a general (non-radially symmetric) profile, the contribution to the Rayleigh integral of the non-radially symmetric part of  $v$  vanishes at any on-axis point. Hence, for a general profile  $v$ , the radially symmetric part of  $v$ , rather than  $v$  itself, is recovered by the method.

## II. SET-UP, NOTATIONS AND BASIC FORMULAS

In this section, the basic formulas as well as the set-up and notations are presented. The radiated pressure is given by the Rayleigh integral as

$$p(\underline{r}, t) = \frac{i\rho_0 ck}{2\pi} e^{i\omega t} \int_S v(\underline{r}_s) \frac{e^{-ikr'}}{r'} dS, \quad (1)$$

where  $\rho_0$  is the density of the medium,  $c$  is the speed of sound in the medium,  $k = \omega/c$  is the wave number and  $\omega$  is the radian frequency of the harmonically vibrating surface  $S$ , in this case a disk of radius  $a$ . Furthermore,  $t$  is time,  $\underline{r}$  is a field point,  $\underline{r}_s$  is a point on the surface  $S$ ,  $r' = |\underline{r} - \underline{r}_s|$  is the distance between  $\underline{r}$  and  $\underline{r}_s$ , and  $v(\underline{r}_s)$  is the normal component of a radially symmetric velocity profile on the surface  $S$ . The time variable  $t$  in  $p(\underline{r}, t)$  and the harmonic factor  $\exp(i\omega t)$  in front of the integral in Eq. (1) will be omitted in the sequel. The average velocity  $V_s$  is given by

$$V = \int_S v(\underline{r}_s) dS = V_s \pi a^2. \quad (2)$$

See Fig. 1 for geometry and notations.

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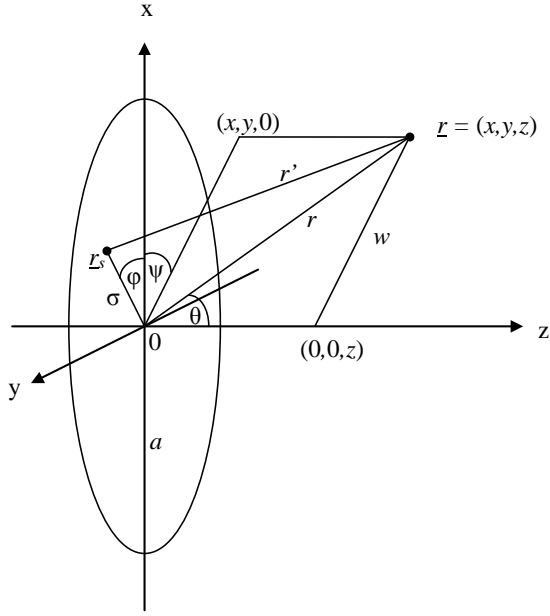


FIG. 1. Set-up and notations. The piston is surrounded by an infinite rigid baffle.

$$\begin{aligned}
r_s &= (x_s, y_s, 0) = (\sigma \cos \varphi, \sigma \sin \varphi, 0) \\
\mathbf{r} &= (x, y, z) = (r \sin \theta \cos \psi, r \sin \theta \sin \psi, r \cos \theta) \\
w &= r \sin \theta = (x^2 + y^2)^{1/2}, \quad z = r \cos \theta \\
r &= |\mathbf{r}| = (x^2 + y^2 + z^2)^{1/2} = (w^2 + z^2)^{1/2} \\
r' &= |\mathbf{r} - \mathbf{r}_s| = (r^2 + \sigma^2 - 2\sigma w \cos(\psi - \varphi))^{1/2}.
\end{aligned}$$

The velocity profiles  $v(\mathbf{r}_s)$  considered are radially symmetric and are denoted as  $v(\sigma)$ ,  $\sigma \geq 0$ , and vanish for  $\sigma > a$ . When such a  $v$  is square integrable over the disk, there is the representation

$$v(\sigma) = V_s \sum_{n=0}^{\infty} u_n R_{2n}^0(\sigma/a), \quad 0 \leq \sigma \leq a, \quad (3)$$

in which the  $u_n$  are scalar coefficients, with  $u_0 = 1$ , and

$$R_{2n}^0(\rho) = P_n(2\rho^2 - 1), \quad 0 \leq \rho \leq 1, \quad (4)$$

are radially symmetric Zernike functions with  $P_n$  the Legendre polynomial<sup>8</sup> (Ch.22) of degree  $n$ . For more details, results and motivation concerning the Zernike terms  $R_{2n}^0$  as expansion functions in the acoustical context, see<sup>9-11</sup>.

There holds<sup>9</sup> for an on-axis point  $\mathbf{r} = (0, 0, z)$  with  $z \geq 0$  the formula

$$p(\mathbf{r}) = \frac{1}{2} \rho_0 c V_s (ka)^2 \sum_{n=0}^{\infty} \gamma_n(k, z) u_n, \quad (5)$$

in which

$$\gamma_n(k, z) = (-1)^n j_n(kz_-) h_n^{(2)}(kz_+), \quad (6)$$

with  $z_{\pm} = \frac{1}{2}((z^2 + a^2)^{1/2} \pm z)$  and  $j_n$  and  $h_n = j_n - iy_n$  the spherical Bessel and Hankel function, respectively, of order  $n = 0, 1, \dots$ , see<sup>8</sup> (§10.1).

The radiated pressure  $p(\mathbf{r})$  (recall Fig. 1 for the notations) is also given by King's integral<sup>12</sup> as

$$p(\mathbf{r}, \omega) = i\rho_0 c k \int_0^{\infty} \frac{e^{-z(u^2 - k^2)^{1/2}}}{(u^2 - k^2)^{1/2}} J_0(wu) V(u) u du, \quad (7)$$

where

$$(u^2 - k^2)^{1/2} = \begin{cases} i\sqrt{k^2 - u^2} & , \quad 0 \leq u \leq k, \\ \sqrt{u^2 - k^2} & , \quad k \leq u < \infty, \end{cases} \quad (8)$$

with  $\sqrt{\quad}$  non-negative, and

$$V(u) = \int_0^a J_0(u\sigma) v(\sigma) \sigma d\sigma, \quad u \geq 0, \quad (9)$$

the Hankel transform of  $v$  and  $J_0$  the Bessel function  $J_{\ell}$  of order  $\ell = 0$ , see<sup>8</sup> (Ch. 9). The Hankel transform of  $v(\sigma)$  in Eq. (3) is given by, see<sup>10</sup> (Eq. (10))

$$V(u) = V_s \sum_{n=0}^{\infty} u_n (-1)^n \frac{a}{u} J_{2n+1}(ua). \quad (10)$$

In Sec. III the impulse response  $\Phi_{\delta}$  is considered. It is defined in accordance with<sup>5</sup> (Sec. VI) and<sup>7</sup> as the potential corresponding to an instantaneous volume displacement  $\Delta$  at  $t = 0$  of the piston with velocity profile  $v$ . It is given for  $t \geq 0$  and any field point  $\mathbf{r}$ , see<sup>5</sup> (Sec. VI), by

$$\Phi_{\delta}(t; \mathbf{r}) = \frac{\Delta}{\pi a^2 V_s} \mathcal{L}^{-1} \left[ \frac{p(\mathbf{r}, \omega)}{i\rho_0 \omega} \Big|_{\omega = -is} \right] (t), \quad (11)$$

with  $\mathcal{L}^{-1}$  the the inverse Laplace transform, and where the wave number  $k = \omega/c$  is replaced by  $s/ic$  with  $s$  the Laplace variable. Greenspan<sup>5</sup> (Sec. VI) uses the King integral representation of  $p$  in Eq. (7), together with an identity for the inverse Laplace transform of  $m^{-1} e^{-zm}$ ,  $m = (u^2 - k^2)^{1/2}$ , to obtain

$$\Phi_{\delta}(t; \mathbf{r}) = \frac{c\Delta H(ct - z)}{\pi a^2 V_s} \int_0^{\infty} J_0(u\sigma(t; z)) J_0(uw) V(u) u du, \quad (12)$$

where  $H$  is the Heaviside function as earlier,  $V(u)$  is the Hankel transform of  $v$  as in Eq. (9), and

$$\sigma(t; z) = \sqrt{c^2 t^2 - z^2}, \quad ct \geq z. \quad (13)$$

In the case that  $\mathbf{r}$  is an on-axis point, so that  $w = 0$ , Greenspan<sup>5</sup> shows by using a number of special results for integrals containing products of Bessel functions that

$$\Phi_{\delta}(t; (0, 0, z)) = \frac{c\Delta}{\pi a^2 V_s} v(\sigma(t; z)) H(ct - z), \quad (14)$$

for the special profiles  $v$

$$v(\sigma) = v^{(n)}(\sigma) = (n+1) V_s (1 - \sigma^2/a^2)^n H(a - \sigma) \quad n = 0, 1, \dots. \quad (15)$$

Here we may note that the right-hand side of Eq. (14) vanishes when  $\sigma(t; z) > a$ . Since any radially symmetric profile  $v$  (vanishing for  $\sigma > a$ ) can be approximated by linear combinations of the special profiles in Eq. (15), it follows that Eq. (14) holds for general velocity profiles vanishing for  $\sigma > a$ . Thus, in general, the velocity profile is reproduced as a warped time-function on the time interval  $[z/c, (z^2 + a^2)^{1/2}/c]$  in which the two limits of the interval correspond to  $\sigma(t; z) = 0$  and  $a$ , respectively.

From the relation

$$ct = (z^2 + \sigma^2(t; z))^{1/2}, \quad (16)$$

it is seen that the values  $v(\sigma)$  of  $v$  are observed at the on-axis point  $(0, 0, z)$  in accordance with the path length between the on-axis point and the points on the piston with distance  $\sigma$  from the origin. Furthermore,

$$\frac{d}{dt}\sigma(t; z) = \frac{ct}{\sigma(t; z)}. \quad (17)$$

Hence, the relative time that  $v(\sigma(t; z))$  spends at a particular value  $v(\sigma)$  with  $0 < \sigma < a$ , is given by

$$\frac{1}{ct}\sigma \approx \sigma/z, \quad (18)$$

where the latter approximation holds for  $z$  large compared to  $a$  (say,  $z \geq 2a$ ). This shows that the values  $v(\sigma)$  of  $v$  are observed properly in Eq. (14) in accordance with their relative importance, that is, with a relative observation time proportional to  $\sigma$ .

### III. HOLOGRAPHY USING TRANSIENT RESPONSES

In<sup>9</sup> a holography method was discussed using on-axis near-field pressure data. This method is based on Eq. (5) and a matching approach in which the expansion coefficients  $u_n$  are chosen such that a best match occurs between the measured on-axis pressure and the right-hand side of Eq. (5). In this section a novel method for the retrieval of baffled-piston velocity profiles from on-axis data is proposed. This method is based on Eq. (14) showing that the impulse responses  $\Phi_\delta(t; (0, 0, z))$ , due to an instantaneous volume displacement  $\Delta$  at  $t = 0$  and observed at the on-axis point  $(0, 0, z)$ , reproduces the velocity profile completely in the scaled- and- warped form

$$\frac{c\Delta}{\pi a^2 V_s} v(\sigma(t; z)), \quad \sigma(t; z) = (c^2 t^2 - z^2)^{1/2}, \quad \frac{z}{c} \leq t \leq \frac{1}{c}(z^2 + a^2)^{1/2}. \quad (19)$$

Hence, the profile is reproduced directly, i.e., without intervention of the expansion coefficients  $u_n$  of Eq. (3). Also, any on-axis point  $(0, 0, z)$  can be used, and this offers the opportunity to suppress noise by employing several on-axis points.

### IV. CONCLUSIONS

Zernike polynomials yield an efficient and robust method to describe velocity profiles of flexible sound radiators and a coefficient-based method to retrieve these profiles. Alternatively, the radially symmetric part of the velocity profile on the disk is reproduced in warped form at any on-axis point as an impulse response. This enables one to solve the inverse problem of calculating the actual velocity profile of the radiator using the (measured) impulse response. This offers a new method in present holography to estimate the velocity profile.

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