

Please leave this heading unchanged!

---

## Sound radiation of a non-rigid piston and pole cap compared with loudspeakers

Ronald M. Aarts, Philips Research Europe HTC 36 (WO-02), NL-5656 AE Eindhoven, The Netherlands,  
Also at Technical University Eindhoven Dept. EE, Den Dolech 2, PT3.23, P.O Box 513, NL-5600 MB Eindhoven, The Netherlands.

Augustus J.E.M. Janssen, Technical University Eindhoven, EURANDOM and Department EE, Den Dolech 2, LG-1, P.O Box 513, NL-5600 MB Eindhoven, The Netherlands.

### Abstract

Loudspeakers are often modelled as a rigid piston in an infinite baffle. This model is for real loudspeakers somewhat limited in two ways. One issue is that a loudspeaker is not rigid and a second issue is that a loudspeaker is mostly used in a cabinet. Both issues are addressed here by developing the velocity of the radiator in terms of orthogonal polynomials known from optical diffraction theory as Zernike circle polynomials. Using these polynomials we develop semi-analytic expressions for the sound pressure from the radiator in two different cases: as a flexible flat radiator mounted in an infinite baffle, and as the cap of a rigid sphere. In the latter case the comparison is done not only for the pressure but also for other quantities viz. the baffle-step response, sound power and directivity, and the acoustic center of the radiator. These quantities are compared with those from a real loudspeaker. Finally, in the case of the baffled-piston radiation the spatial impulse response is presented.

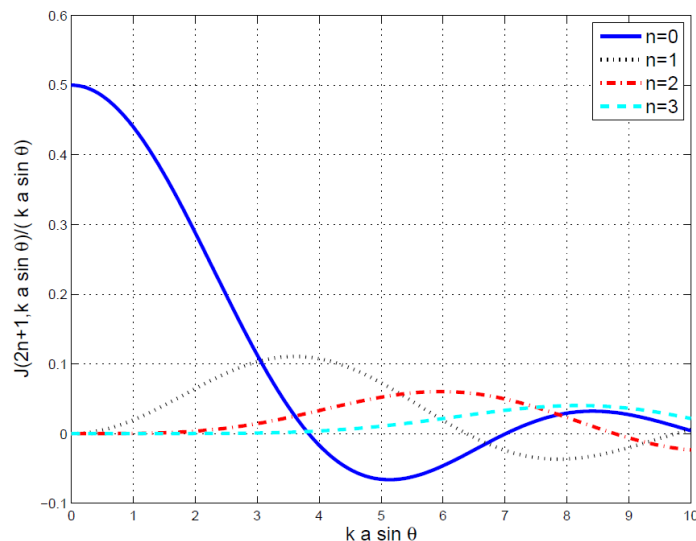
### 1. Nijboer-Zernike approach in acoustics: ANZ

In recent years, the Nijboer-Zernike approach has been applied to solve forward and inverse problems in acoustical radiation from a flexible circular piston surrounded by a rigid infinite planar set (baffle) and from a flexible spherical cap on a rigid sphere [1-11]. The flexibility is embodied by a non-uniform velocity profile  $v$  that is assumed to be radially symmetric in the case of piston radiation and axially symmetric in the case of radiation from the cap. As in optical diffraction theory, the complex amplitude of the sound pressure in the baffled-piston case is given by a Rayleigh integral comprising the field point  $\mathbf{r}$  in front of the baffle and the wave number  $k$  of the harmonic excitation applied to the piston with velocity profile  $v$ . The frequencies that are used in acoustics (20 Hz-20 kHz in audio and up to 200 kHz in ultrasound) are so much lower than those used in optics that dispersion effects can be neglected. Furthermore, although the phenomenon of focusing does manifest itself in acoustics (especially at higher frequencies), this does not occur usually to an extent that one can speak of a focal volume as in optics. Despite all these differences with optics, a wealth of analytic results, with the basic result of the classical Nijboer-Zernike theory as cornerstone, have been obtained for piston radiation and also for radiation from a spherical cap. In both cases, the non-uniform velocity profile is considered to be developed as a series involving the Zernike polynomials (appropriately modified in the case of spherical-cap radiation). The contribution of each of these Zernike terms to the pressure and various other acoustical quantities turns out to have a tractable form, and so the corresponding quantity for the

velocity profile can be obtained in semi-analytic form by linear superposition. Due to the efficiency of the Zernike terms in representing velocity profiles through its expansion coefficients, this offers the opportunity to estimate an unknown velocity profile  $v$  on the level of expansion coefficients from measured data in the field. Here one can use a matching approach in which the unknown expansion coefficients are found by requiring an optimal match between the measured data and the theoretical expression comprising the coefficients.

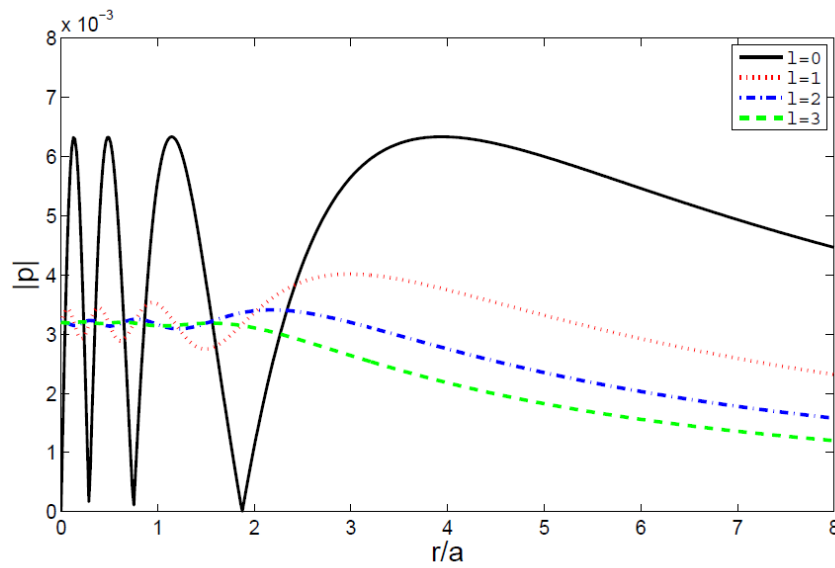
## 2. Far-field pressure obtained from near-field measurements

In Fig. 1 below we have plotted the analytic expression for the far-field pressure in case of baffled-piston radiation due to the first four radially symmetric Zernike terms. The wave number  $k$  of the applied harmonic excitation and the angle  $\theta$  between the acoustical axis (perpendicular to the baffle and passing through the center of the circular piston) and the line segment connecting piston center and field point  $\mathbf{r}$ , have been combined into the single variable  $ka \sin \theta$  (a piston radius).



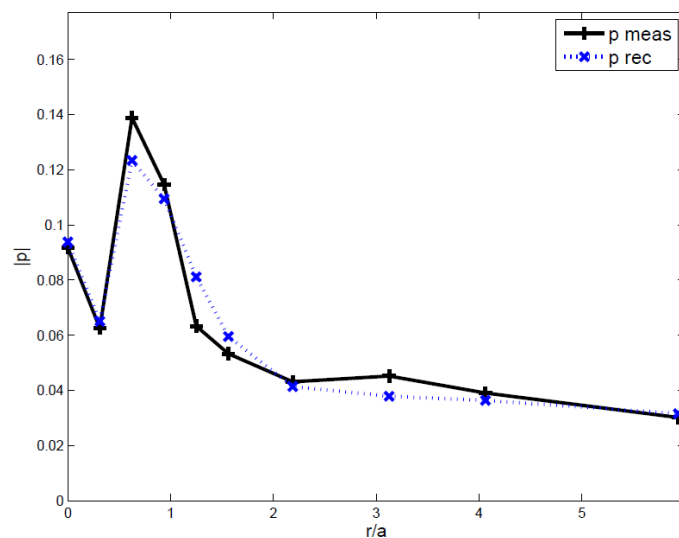
**Figure 1:** The far field pressure  $J_{2n+1}(ka \sin \theta)/(ka \sin \theta)$  as a function of wave number  $k$ , piston radius  $a$  and fieldpoint angle  $\theta$ , combined into the single variable  $ka \sin \theta$ , for the first four Zernike terms (radially symmetric).

In the case that we have a non-uniform velocity profile that can be accurately represented as a linear combination of the first four radially symmetric Zernike terms, one can thus compute the corresponding far-field response by linear super-position. However, the required coefficients of such a linear combination are generally not available. It turns out that these coefficients can be estimated from measured on-axis pressure data. The key result here is an analytic expression, per radially symmetric Zernike term, for the on-axis pressure [1,3,4]. In Fig. 2 the modulus of the normalized on-axis pressure as a function of the normalized on-axis distance is plotted for the case of the first four radially symmetric Zernike terms. The procedure then consists of estimating the unknown expansion coefficients of the piston velocity profile by



**Figure 2:** The modulus of the normalized on-axis pressure  $\frac{1}{2}(l+1) |j_l(kr_-)h_l^{(2)}(kr_+)|$  with  $r_{\pm} = \frac{1}{2}(r^2 + a^2)^{\frac{1}{2}} \pm 1/2r$  as a function of the normalized distance  $r/a$  for the first four radially symmetric Zernike terms.

matching the measured on-axis pressure data with the theoretical expression for the on-axis pressure comprising the unknown expansion coefficients. This procedure has been applied to a real loudspeaker. Thus, the on-axis pressure was measured at 10 near-field points. From these near-field data, the first four coefficients of the radially symmetric Zernike terms describing the velocity profile on the flexible membrane of the loudspeaker were estimated by matching. These four coefficients were fed into the theoretical linear superposition expression for the on-axis pressure [2,3], and the result is displayed in Fig. 3. In this figure the 10 measured on-axis pressure values at normalized axial distance are shown as the solid line connecting the data points, and the estimated on-axis pressure arising from the four estimated coefficients by linear superposition is shown as the dotted line.

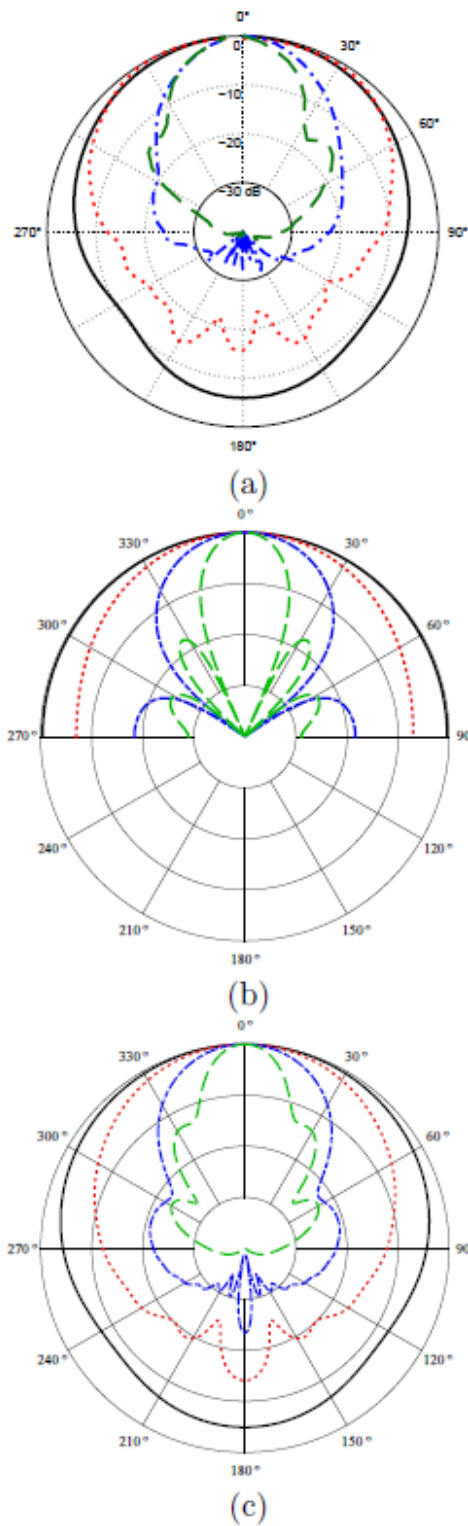


**Figure 3:** measured on-axis near-field pressure data points connected by the black solid line  $p_{\text{meas}}$ , and the estimated on-axis pressure  $p_{\text{rec}}$ , blue dotted line, as a function of the normalized distance  $r/a$ . The loudspeaker is a Vifa MG10 SD09-08 with membrane radius  $a=3.2$  cm and measured in an IEC-baffle at 13.72 kHz.

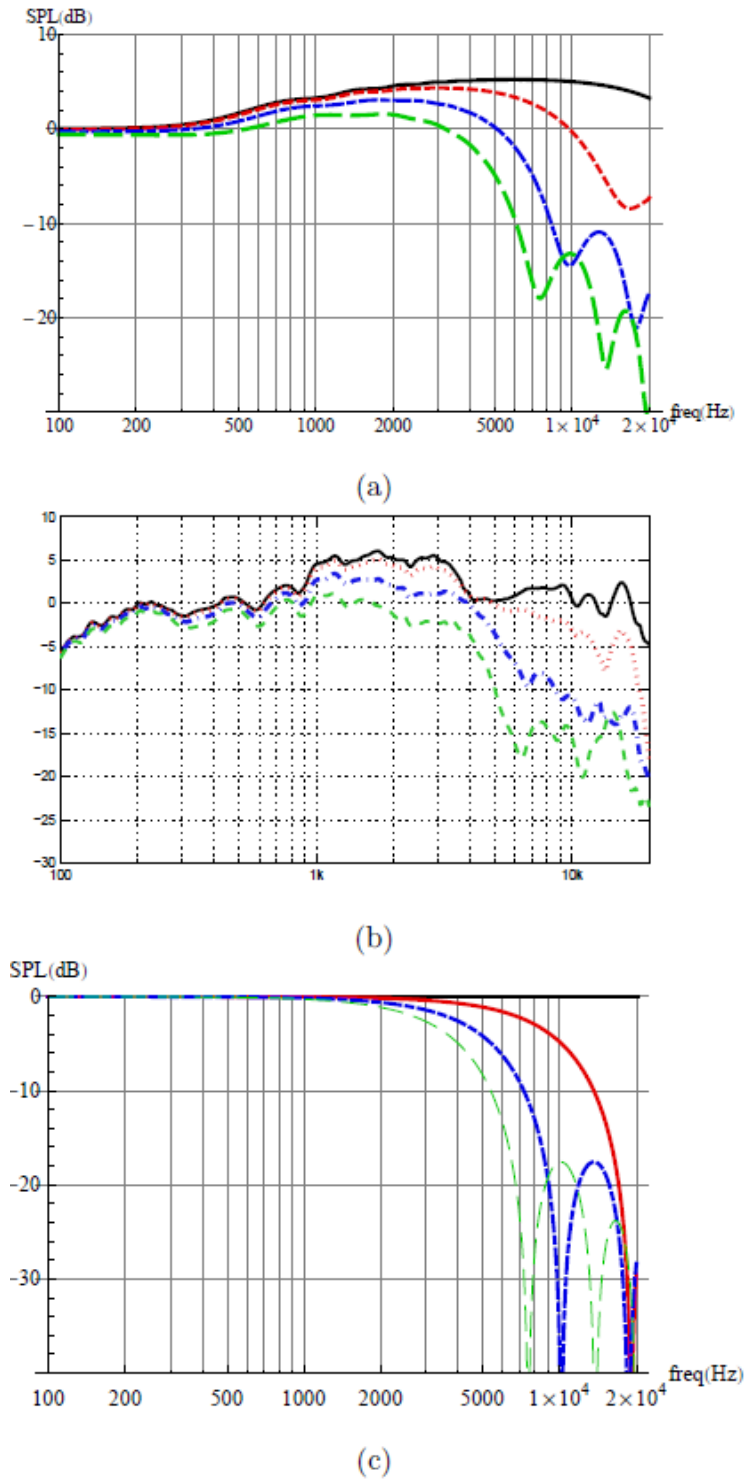
The far-field of the loudspeaker can now also be predicted by linear superposition, using the four estimated coefficients, of the four far-field responses displayed in Fig. 1. Since the measurements were done in the near-field of the loudspeaker, an expensive measurement in the far-field of the loudspeaker by using an anechoic room has been avoided.

### **3. Comparing radiation from a loudspeaker and from a flexible spherical cap on a rigid sphere**

It has been suggested by the theoretical physicists Morse and Ingard that the sound radiation of a loudspeaker in a box is comparable with that of a spherical cap on a rigid sphere when the volumes of box and sphere and the areas of the vibrating membrane and cap are matched. This has been established recently by Aarts and Janssen [7,9,11] who developed, starting from the series solution of the Helmholtz equation with spherical boundary conditions, a Zernike-based computation scheme for the sound pressure on and around the sphere. Thus, the velocity profile on the cap is expanded in Zernike terms (appropriately modified so as to take account of the spherical, rather than flat, nature of the moving cap), and the contribution to the pressure of any field point of each of the Zernike terms is determined in semi-analytic form. In Fig. 4 three polar plots of the modulus of the pressure at four different frequencies and at one meter distance have been compared. The top figure shows the polar plot of a real loudspeaker (the same loudspeaker as the one used in Fig. 3). The middle figure shows the polar plot that arises in the case of baffled-piston radiation with a rigid piston ( $v$  is constant on the piston), and for this the analytic result shown in Fig. 1,  $n=0$ , has been used. The lower figure shows the polar plot of a flexible spherical cap in a rigid sphere with a velocity profile on the cap such that the component in the  $z$ -direction is constant, and for this the Aarts-Janssen computation scheme has been used. The area of the piston and the area of the cap and the volume of the sphere have been chosen the same as the corresponding quantities of the real loudspeaker. As is apparent, the polar plots of the real loudspeaker and those obtained from the spherical-cap model resemble one another much more than the polar plots from the real loudspeaker and those obtained from the baffled-piston model. This remarkable agreement between results for the sphere model and the true loudspeaker continues to hold for several other acoustical quantities such as the baffle-step response, the power and directivity, and the acoustical center [7,11]. At low frequencies the baffle of a loudspeaker is small compared to its wavelength and radiates due to diffraction effects in the full space ( $4\pi$ -field). At those low frequencies the radiator does not benefit from the baffle in terms of gain. At high frequencies the loudspeaker benefits from the baffle which yields a gain of 6 dB. This transition is called the baffle step and is illustrated in Fig. 5 below.

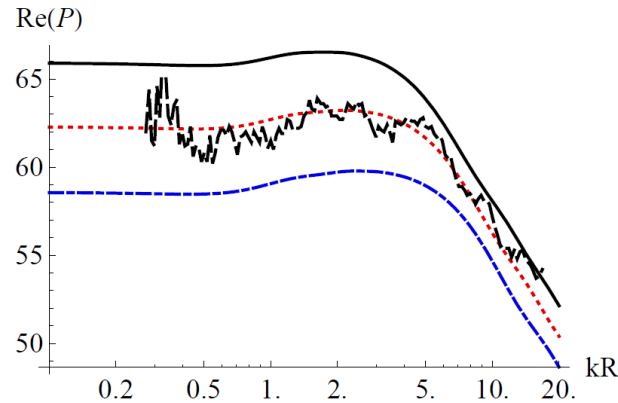


**Figure 4:** Polar plots of the pressure at frequency 1 kHz (solid, black curves), 4 kHz (dotted, red curves), 8 kHz (dashed-dotted, blue curves), and 16 kHz (dashed, green curves), normalized such that the pressure is 1 at  $\theta=0$  for a) loudspeaker (same as in Fig. 3) in a rectangular cabinet measured at 1 m distance, b) rigid piston in an infinite baffle, piston radius  $a=3.2$  cm, using far-field response of Fig. 1,  $n=0$ , c) rigid spherical cap with  $z$ -component of the velocity by profile equal to 1 m/s, cap aperture  $\theta_0=\pi/8$ , and  $r=1$  in the Aarts-Janssen scheme [9].



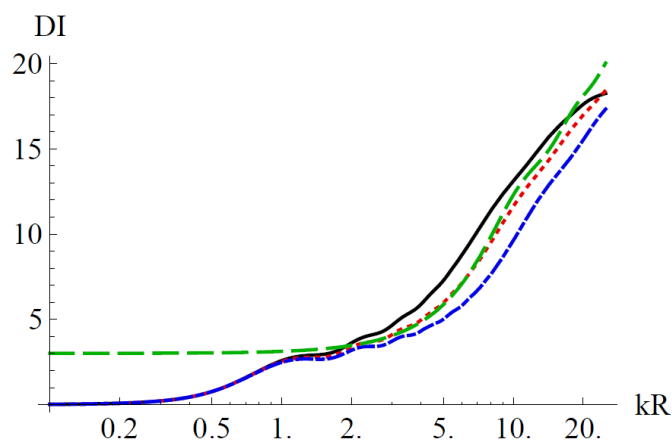
**Figure 5:** Frequency responses for  $\theta = 0$  (solid curve),  $\theta = \pi/9$  (dotted curve),  $\theta = 2\pi/9$  (dashed-dotted curve), and  $\theta = 3\pi/9$  (dashed curve). (a) Baffle step of a polar cap  $\theta_0 = \pi/8$  on a sphere of radius  $R = 0.082$  m, at distance  $r = 1$  m, (constant velocity  $V = v_0 = 1$  m/s). All curves are normalized such that the SPL is 0 dB at 100 Hz. (b) Frequency response of a driver (same as in Fig. 3,4), radius  $a = 3.2$  cm mounted in a square side of a rectangular cabinet with dimensions  $13 \times 13 \times 18.6$  cm. The loudspeaker was measured in an anechoic room at 1 m distance. The on-axis response was normalized to 0 dB at 200 Hz, the other curves were normalized by the same amount. (c) Response of a rigid piston ( $a = 3.2$  cm) in an infinite baffle in the far field. All curves are normalized such that the SPL is 0 dB at 100 Hz.

Figure 6 shows plots of the calculated power for a rigid spherical cap moving with a constant acceleration and various apertures,  $\theta_0=5\pi/32$  (solid curve),  $\theta_0=\pi/8$  (dotted curve), and  $\theta_0=\pi/10$  (dashed-dotted curve), together with the power obtained from the measured loudspeaker (dashed-irregular curve) [7,11].



**Figure 6:** The power  $\text{Re}[P]c/(2\pi\rho_0 a^2 R^4)$  [dB] vs.  $kR$  (log. axis) of a rigid spherical cap moving with a constant acceleration ( $V=i\kappa cV$ ) and various apertures,  $\theta_0=5\pi/32$  (solid curve),  $\theta_0=\pi/8$  (dotted curve), and  $\theta_0=\pi/10$  (dashed-dotted curve), sphere radius  $R=8.2$  cm, together with the power from the measured loudspeaker (same as in Fig. 3-5) (dashed-irregular curve). The logarithmic horizontal axis runs from  $kR=0.1$ -20, corresponding to a frequency range from 66 Hz-13.2 kHz.

In Fig. 7 the directivity index DI from the spherical cap model and the loudspeaker are compared. This directivity index is a qualitative measure of how directive a particular sound radiator is: it is the  $10 \log_{10}$  of the ratio of the modulus squared pressure on the axis in the far field produced by the considered radiator and the same quantity produced by a completely non-directive, point- source radiator, provided that the total radiated power of both radiators is the same [7,11].

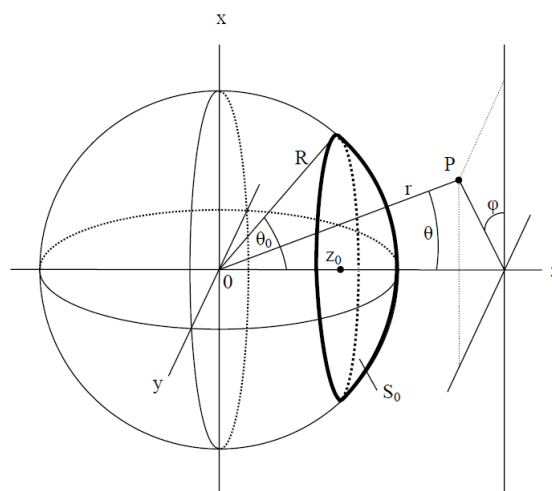


**Figure 7:** The directivity index DI (explained in the text) of a rigid spherical cap with various aperture angles of the cap:  $5\pi/32$  rigid (solid, black curve),  $\pi/8$  (dotted, red curve), and  $\pi/10$  (dashed-dotted, blue curve), The long-dashed, green curve starting for  $kR=0$  at 3 (dB) is the directivity for a rigid piston in an infinite baffle, using the far-field response in Fig. 1,  $n=0$ .

#### 4. Acoustical spatial impulse response

In the case of baffled-piston radiation, the complex amplitude  $p(\mathbf{r};k)$  of the pressure at a field point  $\mathbf{r}$  in front of the baffle due to a harmonic excitation of frequency  $\omega=kc$  with  $c$  the speed of sound is given by one of the Rayleigh integrals [6,10].

Since the media are not dispersive at the relevant frequencies, this Rayleigh integral representation holds in the same form for all involved wave numbers  $k$ . In accordance with well-established practices in physical signal analysis, the spatial impulse response  $\Phi(\mathbf{r}; t)$  can be obtained by performing the Fourier transform of the velocity potential associated with  $p(\mathbf{r}; k)$  with respect to  $k$  in which  $t$  is the Fourier variable. This  $\Phi(\mathbf{r}; t)$  is the impulse response at time  $t > 0$  and at the field point  $\mathbf{r}$  due to an instantaneous volume displacement at  $t=0$  of the piston with non-uniform velocity profile  $v$  vanishing outside the piston, see Fig. 8 below.



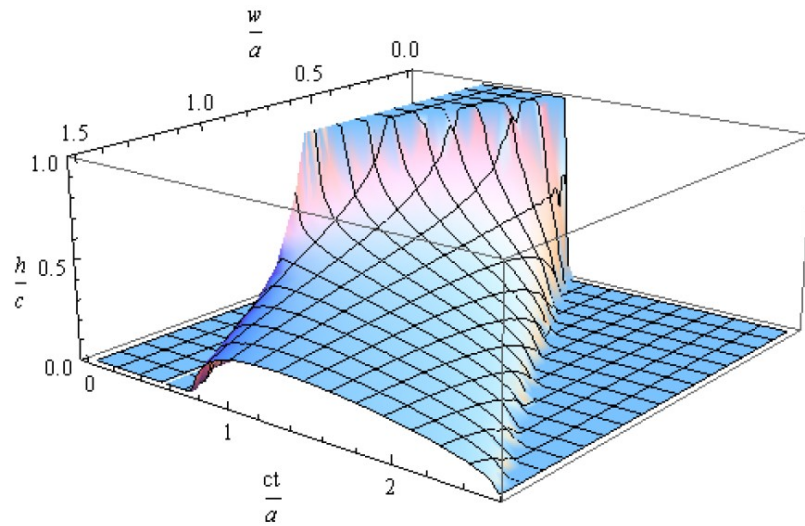
**Figure 8:** Geometry and notations of the pole cap  $s_0$ .

According to the impulse response principle of acoustics, this  $\Phi(\mathbf{r}; t)$  can be obtained as the integral of the velocity profile  $v$  along the arc consisting of all points on the piston that have equal distance  $ct$  to the field point  $\mathbf{r}$ . As a consequence,  $\Phi(\mathbf{r}; t)$  vanishes when  $ct$  is so small or so large that the arc is non-existing or does not contain piston points. In the case that  $v=v(\sigma)$ ,  $0 \leq \sigma \leq a$ , is a radially symmetric profile on the circular piston of radius  $a$  and  $\mathbf{r}=(0, 0, z)$  is an on-axis point, the value of the impulse response  $\Phi(\mathbf{r}=(0, 0, z); t)$  is a multiple of  $v(\sigma=(c^2t^2-z^2)^{1/2})$  when  $0 < c^2t^2-z^2 < a^2$  and 0 otherwise. Hence the radially symmetric profile  $v(\sigma)$ ,  $\sigma \leq 0 \leq a$ , is reproduced at each axial point  $(0, 0, z)$  in warped form where the warping takes places according to  $\sigma=(c^2t^2-z^2)^{1/2}$ . This yields an approach to estimating velocity profiles from on-axis impulse response data.

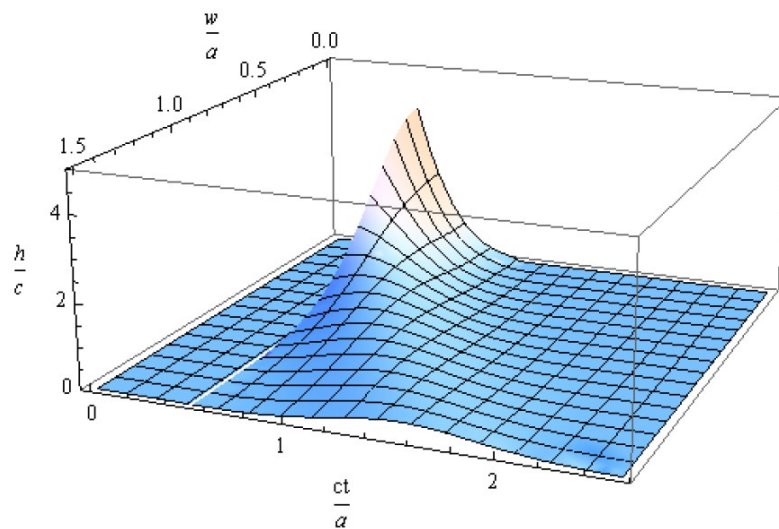
In the case that  $\mathbf{r}$  is a general non-axial, point, the resulting integral expression for the impulse response  $\Phi(\mathbf{r}; t)$  is more complicated, even when  $v$  is radially symmetric. Now it turns out that the approach of developing  $v$  into a Zernike series provides the solution to the forward computation problem. Indeed, Aarts and Janssen [6,10] have shown that each of the Zernike terms involved in the expansion of  $v$  has an explicit, finite-terms expansion for the contribution to the impulse response. In Fig. 9, two impulse responses  $\Phi(\mathbf{r}=(w, 0, z); t)$  are shown, for the case that the normalized axial distance  $z/a$  from the piston plane equals  $1/2$ , as a function of the normalized radial distance  $w/a$  from the axis and the normalized time variable  $ct/a$ . Figure 9 (a) shows this impulse response for the case that  $v$  is constant on the piston (rigid piston) and Fig. 9 (b) shows the impulse response for the case that  $v(\sigma)=\exp(-4(\sigma/a)^2)$ ,



$0 \leq \sigma \leq a$ . The latter velocity profile can be accurately represented using nine radially symmetric Zernike terms, and each of these Zernike terms has a finite-series expression for its contribution to the impulse response.



(a)



(b)

**Figure 9:** Spatial impulse response  $\Phi(w, 0, z; t)$  with constant value  $z/a=1/2$  of the axial variable as a function of the normalized radial variable  $w/a$  and the normalized time variable  $ct/a$  for the cases that (a)  $v(\sigma)=1$ ,  $0 \leq \sigma \leq a$ , (b)  $v(\sigma)=\exp(-4(\sigma/a)^2)$ .

The forward computation method given above can also be used in the reverse direction in which unknown velocity profiles are estimated on the level of their expansion coefficients by matching impulse response data that are measured with the theoretical expression comprising the unknown expansion coefficients.

## 5. Conclusions

Appropriately warped Legendre polynomials provide an efficient and robust method to describe velocity profiles of a non-rigid piston in a baffle and a flexible spherical cap on a rigid sphere. Only a few coefficients are necessary to approximate various velocity profiles. The polar plot of a rigid spherical cap on a rigid sphere has been shown to be quite similar to that of a real loudspeaker, and is useful in the full  $4\pi$ -field. The cap model can be used to predict, besides polar plots, various other acoustical quantities of a loudspeaker including the sound pressure, baffle-step response, sound power, directivity, and the acoustic center. The method enables one to solve the inverse problem of calculating the actual velocity profile of the cap radiator using (measured) on- and off-axis sound pressure data. This computed velocity profile allows the extrapolation to far-field loudspeaker pressure data, including off-axis behavior. Using Zernike expansions of radially symmetric velocity profiles on a baffled, circular piston, a computation scheme for spatial impulse responses is feasible.

## References

The papers listed below are accessible through the following two websites.

<http://www.extra.research.philips.com/hera/people/aarts/>

<http://www.univie.ac.at/nuhag-php/janssen/>

1. R.M. Aarts and A.J.E.M. Janssen, 2009, Estimating the velocity profile and acoustical quantities of a harmonically vibrating loudspeaker membrane from on-axis pressure data, Convention Paper 7678, presented at the 126th Audio Eng. Soc. Convention 2009 May 7-10 Munich, Germany.
2. R.M. Aarts and A.J.E.M. Janssen, 2009, Estimating the velocity profile and acoustical quantities of a harmonically vibrating membrane from on-axis pressure data, Proceedings NAG/DAGA conference, March 23-26 2009, Rotterdam, The Netherlands.
3. R.M. Aarts and A.J.E.M. Janssen, 2009, Estimating the velocity profile and acoustical quantities of a harmonically vibrating loudspeaker membrane from on-axis pressure data, *J. Audio Eng. Soc.*, 57(12):1004-1015.
4. R.M. Aarts and A.J.E.M. Janssen, 2009, On-axis and far-field sound radiation from resilient flat and dome-shaped radiators, *J. Acoust. Soc. Am.*, 125(3):1444-1455.
5. R.M. Aarts and A.J.E.M. Janssen, 2009, Sound radiation quantities arising from a resilient circular radiator, *J. Acoust. Soc. Am.*, 126(4):1776-1787.
6. R.M. Aarts and A.J.E.M. Janssen, 2010, Acoustic holography for piston sound radiation with non-uniform velocity profiles, 17th International Congress on Sound & Vibration, pp. 1-6, Cairo, 18-22 July 2010.
7. R.M. Aarts and A.J.E.M. Janssen, 2010, Modeling a loudspeaker as a resilient spherical cap on a rigid sphere, Convention Paper 7989, presented at the 128th AES Convention 2010 May 22-25 London, UK.
8. R.M. Aarts and A.J.E.M. Janssen, 2010, Authors' Reply on comments on 'Estimating the velocity profile and acoustical quantities of a harmonically vibrating loudspeaker membrane from on-axis pressure data', *J. Audio Eng. Soc.*, 58(4):308-310.
9. R.M. Aarts and A.J.E.M. Janssen, 2010, Sound radiation from a resilient spherical cap on a rigid sphere, *J. Acoust. Soc. Am.*, 127(4):2262-2273.
10. R.M. Aarts and A.J.E.M. Janssen, 2011, Spatial impulse responses from a flexible baffled circular piston, *J. Acoust. Soc. Am.*, 129(4).
11. R.M. Aarts and A.J.E.M. Janssen, 2011, Comparing sound radiation from a loudspeaker with that from a flexible spherical cap on a rigid sphere, to appear in the April 2011 issue of *J. Audio Eng. Soc.*