Classroom proof of the density theorem for Gabor systems

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Abstract.
In this note we present a short and easy proof of the density theorem for Gabor systems (g, a, b):

(i) when (g, a, b) is a Gabor frame we have $ab \leq 1$,

(ii) when (g, a, b) is a Riesz-Gabor basis we have $ab \geq 1$.

1 Introduction

Let $g \in L^2(\mathbb{R})$, $a > 0$, $b > 0$. We denote

$$(g, a, b) = (g_{na,mb})_{n,m \in \mathbb{Z}} \ , \tag{1}$$

and call this a Gabor system, where for $f \in L^2(\mathbb{R})$ and $x, y \in \mathbb{R}$ we use the notation

$$f_{x,y}(t) = e^{2\pi i y t} \, f(t - x) \ , \qquad t \in \mathbb{R} \ , \tag{2}$$

for the time-frequency translate of f over distance (x, y). Gabor systems have been studied extensively the last 15 years, the interest coming from signal theorists and communication theorists. The interest from signal theorists is motivated by the question whether it is possible to expand any signal $f \in L^2(\mathbb{R})$ as

$$f = \sum_{n,m} c_{nm} \, g_{na,mb} \tag{3}$$

in a stable manner with (possibly non-unique) $l^2(\mathbb{Z}^2)$ coefficients c_{nm}. Stated somewhat imprecisely, the signal theorists ask when the system (g, a, b) spans $L^2(\mathbb{R})$ in a decent manner. The interest from communication theorists is motivated by the question whether it is possible to retrieve any data set $(d_{nm})_{n,m\in\mathbb{Z}}$ uniquely from the data-modulated signal

$$\sum_{n,m} d_{nm}\, g_{na,mb} \; . \tag{4}$$

Stated somewhat imprecisely, the communication theorists ask when the system (g, a, b) is sufficiently linearly independent. One should expect here that one cannot answer the signal theorists' question in the affirmative when a and b are large, and that one cannot answer the communication theorists' question in the affirmative when a and b are small.

In modern Gabor theory, these two questions have been brought into a precise mathematical form, viz. whether the system (g, a, b) is a Gabor frame or a Riesz-Gabor basis for the signal theorists and communication theorists, respectively. Furthermore, the following result has been obtained in this theory.

Density theorem for Gabor systems

(i) when (g, a, b) is a Gabor frame, then $ab \le 1$,

(ii) when (g, a, b) is a Riesz-Gabor basis, then $ab \ge 1$.

In Sec. 2 we shall present definitions and preliminaries about Gabor frames and Riesz-Gabor bases. In Sec. 3 we present a proof of the density theorem, based on the unitarity property of the short-time Fourier transform. This proof is short and easy, and can therefore be included in a graduate course on signal theory, communication theory and modern applied Fourier analysis. In Sec. 4 we comment on the underlying duality principle for Gabor systems according to which (g, a, b) is a Gabor frame if and only if $(g, \frac{1}{b}, \frac{1}{a})$ is a Riesz-Gabor basis.

2 Preliminaries from Gabor analysis

There are nowadays excellent textbooks covering Gabor analysis, see [1], Ch. 4 and Secs. 3.4–5, [2] and [3] (advanced Gabor theory and applications), [4], Chs. 5–8 and 11–13, [5], Chs. 5–10. We briefly present the main points here. We call (g, a, b) a Gabor frame when there are $A > 0$, $B < \infty$ (frame bounds) such that for all $f \in L^2(\mathbb{R})$

$$A \, \|f\|_{L^2}^2 \leq \sum_{n,m} |(f, g_{na,mb})|^2 \leq B \, \|f\|_{L^2}^2 \ . \tag{5}$$

We call (g, a, b) a Riesz-Gabor basis when (g, a, b) is a Riesz basis for its linear span: there are $C > 0$, $D < \infty$ (basis bounds) such that for all $d = (d_{nm})_{n,m \in \mathbb{Z}} \in l^2(\mathbb{Z}^2)$

$$C \, \|d\|_{l^2}^2 \leq \Big\| \sum_{n,m} d_{nm} \, g_{na,mb} \Big\|_{L^2}^2 \leq D \, \|d\|_{l^2}^2 \ . \tag{6}$$

Assume that (g, a, b) is a Gabor frame with frame bounds $A > 0$, $B < \infty$. Then the frame operator S is defined on $L^2(\mathbb{R})$ by

$$Sf = \sum_{n,m} (f, g_{na,mb}) \, g_{na,mb} \ , \qquad f \in L^2(\mathbb{R}) \ , \tag{7}$$

and is a bounded, positive definite linear operator of $L^2(\mathbb{R})$. When now $f \in L^2(\mathbb{R})$, we have the representation

$$f = S(S^{-1}f) = \sum_{n,m} (S^{-1}f, g_{na,mb}) \, g_{na,mb} \ , \tag{8}$$

with $L^2(\mathbb{R})$ convergence at the right-hand side of (8).

It is here instructive to view the representation in (8) from the point of view of generalized inverses in linear algebra. Assume that we have integers M, N with $M \geq N \geq 1$, and let $x_1, ..., x_M \in \mathbb{C}^N$. The aim is to represent any $y \in \mathbb{C}^N$ as a linear combination of $x_1, ..., x_M$. Thus, letting X be the $N \times M$ matrix with columns $x_1, ..., x_M$, we want to write an arbitrary $y \in \mathbb{C}^N$ as $y = Xc$ with $c \in \mathbb{C}^M$. This is possible for all y if and only if the $N \times N$ matrix XX^H is invertible (X^H is the $M \times N$ conjugate transpose of X). When XX^H is invertible, we have for any $y \in \mathbb{C}^N$

$$y = (XX^H)(XX^H)^{-1} \, y = X\tilde{c} \ , \tag{9}$$

where $\tilde{c} = X^H(XX^H)^{-1} \, y$ does the job. There may be more c's that do the job for this y, but $\tilde{c}$ is special in the sense that it has minimum Euclidean norm among all $c \in \mathbb{C}^N$ with $y = Xc$.

Accordingly, reading $g_{na,mb}$ as column vectors x_k, f as y, S as XX^H and $((S^{-1}f, g_{na,mb}))_{n,m}$ as $\tilde{c}$, we have that for any $c \in l^2(\mathbb{Z}^2)$ with $f = \sum_{n,m} c_{nm} g_{na,mb}$ there holds

$$\sum_{n,m} |(S^{-1}f, g_{na,mb})|^2 \leq \sum_{n,m} |c_{nm}|^2 \ . \tag{10}$$

The frame operator S commutes with all time-frequency shift operators implicitly involved in the right-hand side series in (7). This holds also for S^{-1}, and using the symmetry of S and S^{-1}, we get for the coefficients $\tilde{c}_{nm} = (S^{-1}f, g_{na,mb})$ in the representation (8) of f

$$\tilde{c}_{nm} = (f, S^{-1}g_{na,mb}) = (f, (S^{-1}g)_{na,mb}) = (f, {}^{\circ}\gamma_{na,mb}) \ , \tag{11}$$

where we have set ${}^{\circ}\gamma = S^{-1}g$. We call this ${}^{\circ}\gamma$ the canonical dual corresponding to the Gabor frame (g, a, b). The Gabor system $({}^{\circ}\gamma, a, b)$ is also a Gabor frame, with frame bounds B^{-1}, A^{-1} and frame operator S^{-1}.

Summarizing we have for a Gabor frame (g, a, b) and an $f \in L^2(\mathbb{R})$ the $L^2(\mathbb{R})$-convergent representation

$$f = \sum_{n,m} (f, {}^{\circ}\gamma_{na,mb})\, g_{na,mb} \ , \tag{12}$$

and for any other representation $f = \sum_{n,m} c_{nm}\, g_{na,mb}$ with $c \in l^2(\mathbb{Z}^2)$, we have

$$\sum_{n,m} |(f, {}^{\circ}\gamma_{na,mb})|^2 \le \sum_{n,m} |c_{nm}|^2 \ . \tag{13}$$

Now assume that (g, a, b) is a Riesz-Gabor basis, see (6). We let for $n', m' \in \mathbb{Z}$

$$V = \text{closed linear span of } (g_{na,mb})_{n,m\in\mathbb{Z}} \ , \tag{14}$$

$$V_{n',m'} = \text{closed linear span of } (g_{na,mb})_{n,m\in\mathbb{Z},(n,m)\neq(n',m')} \ . \tag{15}$$

Furthermore, we let $P_{n',m'}$ be the orthogonal projection onto $V_{n',m'}$, and we let

$$h_{n',m'} = \frac{g_{n'a,m'b} - P_{n',m'}\, g_{n'a,m'b}}{\|g_{n'a,m'b} - P_{n',m'}\, g_{n'a,m'b}\|^2} \ . \tag{16}$$

Due to the assumption we have that $\|h_{n',m'}\|$ is bounded between D^{-1} and C^{-1}, and there is the biorthogonality relation

$$(g_{na,mb}, h_{n',m'}) = \delta_{nn'}\, \delta_{mm'} \ , \tag{17}$$

where we have used Kronecker's delta. Hence, when $(d_{nm})_{n,m\in\mathbb{Z}^2} \in l^2(\mathbb{Z}^2)$, we can uniquely retrieve the data d_{nm} from the modulated signal $\sum_{n,m} d_{nm}\, g_{na,mb}$ as

$$d_{n'm'} = \Big(\sum_{n,m} d_{nm}\, g_{na,mb}, h_{n',m'}\Big) \ , \qquad n', m' \in \mathbb{Z} \ . \tag{18}$$

In the picture of generalized inverses from linear algebra, we now consider $x_1, ..., x_M \in \mathbb{C}^N$ with integer M, N such that $1 \le M \le N$, and we let X

be the $N \times M$-matrix with columns $x_1, ..., x_M$ (as before). Now we want a left-inverse Y^H of X, i.e. an $N \times M$-matrix Y such that $Y^H X = I_M$, so that for any $c \in \mathbb{C}^M$ we can retrieve c from Xc as $c = Y^H(Xc)$. This is possible if and only if the $M \times M$-matrix $X^H X$ is invertible, and then a possible choice for Y is $X(X^H X)^{-1}$. This choice is special for the following reason. The columns $y_1, ..., y_M$ of Y all lie in the linear span of X and they satisfy $y_k^H x_l = \delta_{kl}$. Hence y_k can be obtained as a multiple of $x_k - P_k x_k$, where P_k is the orthogonal projector of $\mathbb{C}^N$ onto the linear span of x_l, $l \neq k$.

Since the time-frequency shift operators $f \to f_{na,mb}$ are unitary and commute with the projection operators $P_{n',m'}$, it follows that

$$h_{n',m'} = ({}^{\circ\circ}\gamma)_{n'a,m'b} \tag{19}$$

where we have denoted ${}^{\circ\circ}\gamma = h_{o,o}$. Furthermore, $({}^{\circ\circ}\gamma, a, b)$ is a Riesz-Gabor basis, with closed linear span equal to V and with basis bounds D^{-1}, C^{-1} (see (6)). Finally, when $f \in L^2(\mathbb{R})$, the orthogonal projection Pf of f onto V is given by

$$Pf = \sum_{n,m} (f, {}^{\circ\circ}\gamma_{na,mb})\, g_{na,mb} = \sum_{n,m} (f, g_{na,mb})\, {}^{\circ\circ}\gamma_{na,mb} \ . \tag{20}$$

In particular, there holds for $f \in L^2(\mathbb{R})$

$$\Big|\sum_{n,m} ({}^{\circ\circ}\gamma_{na,mb}, f)(f, g_{na,mb})\Big| = |(f, Pf)| \leq \|f\|^2 \ . \tag{21}$$

3 Proof of the density theorem

Besides the preliminaries in Sec. 2, there is one more ingredient needed in proving the density theorem. This is what is called in physics the resolution of identity, [1], Secs. 2.4 and 2.7, and in time-frequency analysis the orthogonality relation for the short-time Fourier transform, [4], Sec. 3.2, [5], Prop. 8.1.2. It reads as follows. Let $f, h, g, \gamma \in L^2(\mathbb{R})$. Then $(f, \gamma_{x,y})$ and $(h, g_{x,y})$ belong to $L^2(\mathbb{R}^2)$ as functions of $x, y \in \mathbb{R}$, and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \gamma_{x,y})(g_{x,y}, h)\, dx\, dy = (g, \gamma)(f, h) \ . \tag{22}$$

We are now ready to prove the density theorem.

Proof of (i). Assume that (g, a, b) is a Gabor frame. We shall first show that $(g, {}^{\circ}\gamma) = ab$, where ${}^{\circ}\gamma = S^{-1}g$ as in (11). To that end we take $f, h \in L^2(\mathbb{R})$

with $(f,h)\neq 0$. Now for any $x,y\in\mathbb{R}$ we have from (12) and the properties of the time-frequency shift operators that

$$(f,h)=(f_{-x,-y},h_{-x,-y})=\sum_{n,m}(f,{}^{\circ}\gamma_{na+x,mb+y})(g_{na+x,mb+y},h)\ . \qquad (23)$$

Integrating this identity over $(x,y)\in[0,a)\times[0,b)$, we find

$$ab(f,h) \;=\; \int_0^a\int_0^b \sum_{n,m}(f,{}^{\circ}\gamma_{na+x,mb+y})(g_{na+x,mb+y},h)\,dx\,dy \;=$$

$$=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(f,{}^{\circ}\gamma_{x,y})(g_{x,y},h)\,dx\,dy=(g,{}^{\circ}\gamma)(f,h)\ . \qquad (24)$$

Here we have used dominated convergence, also see (5), and (22). Hence $(g,{}^{\circ}\gamma)=ab$ since $(f,h)\neq 0$.

Next consider the two representations

$$g=\sum_{n,m}(g,{}^{\circ}\gamma_{na,mb})\,g_{na,mb}=1\cdot g+\sum_{(n,m)\neq(0,0)}0\cdot g_{na,mb}\ . \qquad (25)$$

Then (13) applied with $f=g$ yields

$$\sum_{n,m}|(g,{}^{\circ}\gamma_{na,mb})|^2\leq 1^2+\sum_{(n,m)\neq(0,0)}0^2=1\ . \qquad (26)$$

Hence $ab=(g,{}^{\circ}\gamma)=(g,{}^{\circ}\gamma_{o,o})\leq 1$, as required.

Proof of (ii). Assume that (g,a,b) is a Riesz-Gabor basis. Take $f\in L^2(\mathbb{R})$ with $\|f\|=1$. By (17), applied with $n=m=0=n'=m'$, and (19) and (22), we have

$$1 \;=\; ({}^{\circ\circ}\gamma,g)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}({}^{\circ\circ}\gamma,f_{x,y})(f_{x,y},g)\,dx\,dy \;=$$

$$=\; \int_0^a\int_0^b\sum_{n,m}({}^{\circ\circ}\gamma_{na,mb},f_{x,y})(f_{x,y},g_{na,mb})\,dx\,dy \qquad (27)$$

by dominated convergence, see also (21). Applying (21), with $f=f_{x,y}$ and $x,y\in[0,a)\times[0,b)$, we find

$$1\leq\int_0^a\int_0^b\|f_{x,y}\|^2\,dx\,dy=ab\,\|f\|^2=ab\ . \qquad (28)$$

Here it has been used that $\|f_{x,y}\|^2=\|f\|^2$ for $x,y\in\mathbb{R}$ by unitarity of the time-frequency shift operators. Hence $ab\geq 1$, as required.

4 The duality principle for Gabor systems

The notions of Gabor frame and of Riesz-Gabor basis are intimately related through the duality principle for Gabor systems. The following holds. We have that (g, a, b) is a Gabor frame if and only if $(g, \frac{1}{b}, \frac{1}{a})$ is a Riesz-Gabor basis, and the frame bounds A, B in (5) and the basis bounds C, D in (6) are related as $C = ab\,A$, $D = ab\,B$. The result is often referred to as the Ron-Shen duality principle, see [6]. However, as is evidenced by the acknowledgements section, Subsec 1.4 in [6], this principle was developed independently and more or less simultaneously as well in [7] and [8], using methods that are quite different from one another and from those in [6]. The method given above to prove (i) and (ii) is contained in the approach to the duality principle used in [7]. A further result of this duality principle is that ${}^{\circ}\gamma$ and ${}^{\circ\circ}\gamma$ in the proofs of (i) and (ii) are related as ${}^{\circ}\gamma = ab\,{}^{\circ\circ}\gamma$ (here (i) is considered with a, b while (ii) is considered with $\frac{1}{b}$, $\frac{1}{a}$ instead of a, b).

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